Classical Dynamics

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Lockdown 3.0

1 Gravitational Fields

1.1 Polar Coordinates

$$\begin{aligned}
\boldsymbol{r} &= r\hat{\boldsymbol{e}}_r \\
\dot{\boldsymbol{r}} &= \dot{r}\hat{\boldsymbol{e}}_r + r\dot{\phi}\hat{\boldsymbol{e}}_\phi \\
\ddot{\boldsymbol{r}} &= \left(\ddot{r} - r\dot{\phi}^2\right)\hat{\boldsymbol{e}}_r + \left(2\dot{r}\dot{\phi} + r\ddot{\phi}\right)\hat{\boldsymbol{e}}_\phi \\
&= \left(\ddot{r} - r\dot{\phi}^2\right)\hat{\boldsymbol{e}}_r + \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\phi}\right)\hat{\boldsymbol{e}}_\phi
\end{aligned}$$

1.2 General Orbits

For all central forces (i.e $\boldsymbol{F} \propto \boldsymbol{r}$), the torque $\boldsymbol{G} = \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{0}$. Thus the angular momentum $\boldsymbol{J} = \boldsymbol{r} \times \boldsymbol{p}$ is conserved. In polar coordinates, $\boldsymbol{J} = mr^2\dot{\phi}\,\hat{\boldsymbol{e}}_z$. The total energy of a point mass in a central force is therefore:

$$\begin{aligned}
E &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + U(r) \\
&= \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + U(r)
\end{aligned}$$

to reduce the problem to one dimension. The final two terms are often grouped together into a single "effective potential" $U_{\text{eff}}(r)$.

1.3 Inverse Square Orbits

For attractive forces, $U(r) = -A/r$ for some positive constant A. We thus have:

$$E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} - \frac{A}{r}$$

It turns out to be easier to work in $u = 1/r$ and $\dot{r} = \frac{dr}{du}\frac{du}{d\phi}\dot{\phi} = -r^2\dot{\phi}\frac{du}{d\phi} = -\frac{J}{m}\frac{du}{d\phi}$. So

$$\begin{aligned}
\frac{J^2}{2m}\left(\frac{du}{d\phi}\right)^2 + \frac{J^2}{2m}u^2 - Au &= E \\
\left(\frac{du}{d\phi}\right)^2 + \Big(u - \underbrace{\frac{mA}{J^2}}_{1/r_0}\Big)^2 &= \underbrace{\frac{2mE}{J^2} + \left(\frac{mA}{J^2}\right)^2}_{e^2/r_0^2} \\
\left(\frac{du}{d\phi}\right) + \left(u - \frac{1}{r_0}\right)^2 &= \frac{e^2}{r_0^2} \\
\Rightarrow u(\phi) = \frac{1 \pm e\cos(\phi)}{r_0} &= \frac{1}{r(\phi)}
\end{aligned}$$

$$\Rightarrow r(\phi) = \frac{r_0}{1 \pm e\cos(\phi)} \tag{1}$$

This is the famous equation for an orbit. Note the useful variables:

$$\begin{aligned}
r_0 &= \frac{J^2}{mA} \\
e^2 &= 1 + \frac{2mEr_0^2}{J^2} = 1 + \frac{2Er_0}{A} \\
&= 1 + \frac{2EJ^2}{mA^2}
\end{aligned}$$

and note that the former is a function only of J. Also, note that r_0 is the distance from the origin when $\phi = \pi/2$, i.e. the semi-latus rectum. The polar expression (1) can easily be converted into a Cartesian one:

$$\begin{aligned}
r &= r_0 \mp ex \\
x^2 + y^2 &= e^2x^2 \mp 2er_0x + r_0^2 \\
x^2 \pm 2\frac{er_0}{1-e^2} + \frac{y^2}{1-e^2} &= \frac{r_0^2}{1-e^2} \\
\left(x \pm \frac{er_0}{1-e^2}\right)^2 + \frac{y^2}{1-e^2} &= \frac{r_0^2}{1-e^2} + \frac{e^2r_0^2}{(1-e^2)^2}
\end{aligned}$$

$$\left(x \pm \frac{er_0}{1-e^2}\right)^2 + \frac{y^2}{1-e^2} = \frac{r_0^2}{(1-e^2)^2} \tag{2}$$

1.3.1 Circular Orbits

Let $e = 0$. Backtracking to (1), this simply gives $r(\phi) = r_0$, a circle with radius r_0. Alternatively, (2) gives $x^2 + y^2 = r_0^2$. The formula for e gives that circular orbits have:

$$E = -\frac{A}{2r_0}$$

1.3.2 Elliptical Orbits

Let $e < 1$. (2) then becomes:

$$\left(\frac{x \pm \frac{er_0}{1-e^2}}{r_0/(1-e^2)}\right)^2 + \left(\frac{y}{r_0/\sqrt{1-e^2}}\right)^2 = 1$$

This is the formula for an ellipse. If $e < 1$, note that such elliptical orbits have $E < 0$. Note the following properties:

$$\begin{aligned}
a &= \frac{r_0}{1-e^2} \\
b &= \frac{r_0}{\sqrt{1-e^2}} \\
e^2 &= 1 - \frac{b^2}{a^2} \\
r_{\text{min}} &= \frac{r_0}{1+e} = a(1-e) \\
r_{\text{max}} &= \frac{r_0}{1-e} = a(1+e) \\
2a &= r_{\text{min}} + r_{\text{max}} \\
E = -\frac{A(1-e^2)}{2r_0} &= -\frac{A}{2a} = -\frac{A}{r_{\text{min}} + r_{\text{max}}} \\
\text{Area} = \pi ab &= \frac{\pi r_0^2}{(1-e^2)^{3/2}} \\
\text{Area sweep rate} &= \frac{1}{2} r^2 \dot{\phi} = \frac{J}{2m} \\
\Rightarrow T &= \frac{2m\pi r_0^2}{J(1-e^2)^{3/2}} = 2\pi\sqrt{\frac{ma^3}{A}}
\end{aligned}$$

with the last expressing the useful fact that $T^2 \propto a^3$.

1.3.3 Parabolic Orbits

Let $e = 1$. Backtracking to just before (2), we have $y^2 = r_0^2 \pm 2r_0 x$, a parabola. The formula for e clearly gives $E = 0$.

1.3.4 Hyperbolic Orbits

Let $e > 1$. (2) then becomes:

$$\left(\frac{x \mp \frac{er_0}{e^2-1}}{r_0/(e^2-1)}\right)^2 - \left(\frac{y}{r_0/\sqrt{e^2-1}}\right)^2 = 1$$

This is the formula for a hyperbola, but only one of the branches is physical, depending on whether the orbit is attractive or repulsive. As $e > 1$, hyperbolic orbits have $E > 0$. Taking $e > 0$ wlog from now on, it turns out to be usually more useful to use the radial equation (1), to deduce several things:

- There are two asymptotes (for each $e > 0$), at $\cos\phi_\infty = -1/e$. Note that this means $\tan^2\phi_\infty = e^2 - 1$
- Many problems require the total angle of deflection $\chi = 2\phi_\infty - \pi$.
- The impact parameter b is the minimum distance between an asymptote and the origin. It turns out that this quantity is analogous to the semi-minor axis of an elliptical orbit, and is equal to $r_0/\sqrt{e^2-1}$
- If the asymptotic velocity is v_∞, then $J = mbv_\infty$ and $E = \frac{1}{2}mv_\infty^2$

1.3.5 Repulsive Forces

For repulsive inverse-square trajectories (e.g. Rutherford Scattering), the potential energy is positive so $E > 0$ all the time. Generally though, swapping from attractive to repulsive means $A \to -A$, $r_0 \to -r_0$; e is unchanged. Thus the equation of the orbit becomes:

$$r(\phi) = \frac{r_0}{\pm e\cos(\phi) - 1}$$

The physical branch of repulsive hyperbolic orbits happens to be the positive root: $r = r_0/(e\cos(\phi) - 1)$. The point of closest approach is $r_{\min} = r_0/(e - 1) = a/(1+e)$.

The deflection angle χ can be calculated geometrically, but an alternative derivation is possible. The momentum in the direction of approach is $\Delta p = mv_\infty(\cos\chi - 1) = -2mv_\infty\sin^2(\chi/2)$, but also

$$\begin{aligned}\Delta p &= \int_{-\infty}^{\infty} -\frac{A}{r^2}\cos\theta\ dt = -\int_0^{\pi-\chi}\frac{A}{r^2\dot\theta}\cos\theta\ d\theta = -\frac{Am}{J}\int_0^{\pi-\chi}\cos\theta\ d\theta \\ &= -\frac{Am}{J}\sin\chi = -\frac{2Am}{J}\sin(\chi/2)\cos(\chi/2)\end{aligned}$$

$$\Rightarrow \tan(\chi/2) = \frac{A}{Jv_\infty} = \frac{A}{mv_\infty^2 b}$$

1.4 Two-Body Problem

In reality, the heavier body is not fixed and the two bodies both orbit their mutual barycentre, in ellipses with the same eccentricity and phase, though this is not noticeable if one is far heavier than the other. A special case of these systems is where both orbits are circular, with the centre of mass at the centre of each. For two masses M_1 and M_2 displaced by a distance r, the CoM is $M_2 r/(M_1 + M_2)$ away from M_1 and vice versa.

The angular velocity of the orbits is easily found from force balance:

$$\frac{GM_1M_2}{r^2} = M_1\omega^2 \frac{M_2 r}{M_1 + M_2}$$
$$\Rightarrow \omega^2 = \frac{G(M_1 + M_2)}{r^3}$$

A more detailed analysis involves supposing that M_1 is at $\boldsymbol{r_1}$ etc. It then follows that:

$$T = \frac{1}{2}\mu\left(\dot{\boldsymbol{r}}_1 - \dot{\boldsymbol{r}}_2\right)^2$$
$$\boldsymbol{J} = \mu\left(\boldsymbol{r}_1 - \boldsymbol{r}_2\right) \times \left(\dot{\boldsymbol{r}}_1 - \dot{\boldsymbol{r}}_2\right)$$
$$\boldsymbol{F_{12}} = \mu\left(\ddot{\boldsymbol{r}}_1 - \ddot{\boldsymbol{r}}_2\right)$$

thus reducing the two-body problem to a one-body problem, with a reduced mass $\mu = M_1M_2/(M_1 + M_2)$.

1.4.1 Three-Body Problem

There's no general solution for this, but what often happens is that two of the three form a close binary, losing a lot of PE each. The third takes this energy as KE, and is often able to become unbound entirely.

1.5 Tidal Fields

One cannot actually detect locally the gravitational field, but one can detect the difference between the gravitational field and that at a nearby point. One defines the vector field $\boldsymbol{T}(\boldsymbol{r};\boldsymbol{a}) = \boldsymbol{a} \cdot \boldsymbol{\nabla g}$, parametrised by some small vector $\boldsymbol{a}$.

Consider $\boldsymbol{T}(\boldsymbol{r}; a\hat{\boldsymbol{e}}_r)$. Now

$$\boldsymbol{g}(\boldsymbol{r}) = -\frac{GM}{r^2}\hat{\boldsymbol{e}}_r \qquad \begin{aligned} \boldsymbol{g}(\boldsymbol{r} + a\hat{\boldsymbol{e}}_r) &= -\frac{GM}{(r+a)^2}\hat{\boldsymbol{e}}_r \\ &\approx -\frac{GM}{r^2}\hat{\boldsymbol{e}}_r + \frac{2aGM}{r^3}\hat{\boldsymbol{e}}_r \\ \Rightarrow \boldsymbol{T}(\boldsymbol{r}; \hat{\boldsymbol{e}}_r) &= \frac{2GM}{r^3} \end{aligned}$$

Consider $\boldsymbol{T}(\boldsymbol{r}; a\hat{\boldsymbol{e}}_\phi)$. Now

$$\begin{aligned} \boldsymbol{g}(\boldsymbol{r} + a\hat{\boldsymbol{e}}_\phi) &\approx -\frac{GM}{r^2}\hat{\boldsymbol{e}}_r - \frac{GM}{r^2}\frac{a}{r}\hat{\boldsymbol{e}}_\phi \\ \Rightarrow \boldsymbol{T}(\boldsymbol{r}; \hat{\boldsymbol{e}}_\phi) &= -\frac{GM}{r^3}\hat{\boldsymbol{e}}_\phi \end{aligned}$$

and similarly for $\boldsymbol{T}(\boldsymbol{r}; \hat{\boldsymbol{e}}_\theta)$.

If one is rotating around an axis (for instance, standing on the surface of a rotating planet), the centrifugal force also produces a tidal effect. Supposing that the rotation is about the $\theta = 0$ (i.e. the z) axis, the total tidal vectors are:

$$\boldsymbol{T}(\boldsymbol{r}; \hat{\boldsymbol{e}}_r) = \frac{3GM}{r^3}\hat{\boldsymbol{e}}_r \qquad \boldsymbol{T}(\boldsymbol{r}; \hat{\boldsymbol{e}}_\phi) = \boldsymbol{0} \qquad \boldsymbol{T}(\boldsymbol{r}; \hat{\boldsymbol{e}}_\theta) = -\frac{GM}{r^3}\hat{\boldsymbol{e}}_\theta$$

2 Rotation

2.1 Rotating Frames of Reference

Consider a frame S whose unit vectors are all rotating with angular velocity $\boldsymbol{\omega}$ with respect to an inertial frame S_0. As a result, we have for instance:

$$\frac{d}{dt}\hat{\boldsymbol{e}}_z = \boldsymbol{\omega} \times \hat{\boldsymbol{e}}_z$$

For a vector which might be changing in S_0, we therefore have the operator equation:

$$\left[\frac{d}{dt}\right]_{S_0} = \left[\frac{d}{dt}\right]_S + \boldsymbol{\omega}\times$$

Applying this to $\boldsymbol{r}$, we have:

$$\begin{aligned} \left[\frac{d\boldsymbol{r}}{dt}\right]_{S_0} &= \left[\frac{d\boldsymbol{r}}{dt}\right]_S + \boldsymbol{\omega} \times \boldsymbol{r} \\ &= \boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{r} \end{aligned}$$

as expected. Applying it again, we have:

$$\left[\frac{d^2\boldsymbol{r}}{dt^2}\right]_{S_0} = \left[\frac{d\boldsymbol{v}}{dt}\right]_S + \left[\frac{d}{dt}(\boldsymbol{\omega} \times \boldsymbol{r})\right]_S + \boldsymbol{\omega} \times \boldsymbol{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})$$
$$= \boldsymbol{a} + 2\boldsymbol{\omega} \times \boldsymbol{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}$$

Therefore, the acceleration as seen in the rotating frame S (at a given point in time) is:

$$\boldsymbol{a} = \underbrace{\left[\frac{d^2\boldsymbol{r}}{dt^2}\right]}_{S_0} \underbrace{-2\boldsymbol{\omega} \times \boldsymbol{v}}_{\text{Coriolis}} \underbrace{-\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})}_{\text{Centrifugal}} \underbrace{-\dot{\boldsymbol{\omega}} \times \boldsymbol{r}}_{\text{Euler}}$$

where the Coriolis, Centrifugal, and Euler *forces* are the above terms multiplied by m.

2.1.1 Coriolis Force

$$\boldsymbol{F_{cr}} = -2m\boldsymbol{\omega} \times \boldsymbol{v}$$

We see that $\boldsymbol{F_c}$ is perpendicular to both $\boldsymbol{\Omega}$ and $\boldsymbol{v}$. On the surface of the Earth, $\boldsymbol{\Omega}$ is upwards from the North Pole, so when one is travelling along the surface of the Earth at latitude λ, the magnitude of the Coriolis force is $2m\Omega v \sin\lambda$.

Care should be taken when analysing Coriolis effects, as their direction can be counter-intuitive. It should be borne in mind that Coriolis problems can often be analysed instead by conservation of angular momentum, and when one moves closer to the axis of rotation the Coriolis force has the effect of increasing one's angular velocity. As such, if one moves North in the northern hemisphere (i.e. closer to the Earth's axis), the Coriolis Force acts Eastwards. In fact, when travelling Eastwards in the northern hemisphere, the Coriolis Force acts southwards... in other words, the Coriolis force always acts to the *right* when in the northern hemisphere, and the component along the surface of the Earth always has the same magnitude. Guess what it does in the southern hemisphere. Anyway it causes cyclones.

2.1.2 Centrifugal Force

$$\boldsymbol{F_{cf}} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})$$

One can rearrange this to give

$$\boldsymbol{F_{cf}} = m(\omega^2\boldsymbol{r} - (\boldsymbol{\omega} \cdot \boldsymbol{r})\boldsymbol{\omega})$$
$$= m\omega^2(\boldsymbol{r} - (\hat{\omega} \cdot \boldsymbol{r})\hat{\omega})) = m\omega^2\boldsymbol{\rho}$$

where ρ is the perpendicular vector from the rotation axis to the tip of $\boldsymbol{r}$. The centrifugal force gives the Earth an equatorial bulge of about 1/300.

2.1.3 Euler Force

$$\boldsymbol{F_e} = -m\dot{\boldsymbol{\omega}} \times \boldsymbol{r}$$

The Euler force is useless.

2.2 Inertia Tensor

For a rigid body, every element of a body has the same $\boldsymbol{\omega}$. As such, the angular momentum $\boldsymbol{J}$ is given by:

$$\begin{aligned}
\boldsymbol{J} &= \int dm\ \boldsymbol{r} \times \dot{\boldsymbol{r}} \\
&= \int dm\ \boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r}) \\
&= \int dm\ (r^2\boldsymbol{\omega} - (\boldsymbol{r} \cdot \boldsymbol{\omega})\boldsymbol{r}) \\
\Rightarrow J_i &= \int dm\ (x_k x_k \omega_i - x_j \omega_j x_i) \\
&= \int dm\ (x_k x_k \delta_{ij} - x_i x_j)\omega_j \\
&\equiv I_{ij}\omega_j
\end{aligned}$$

where the last line defines the components of the symmetric moment of inertia tensor $\underline{\underline{\boldsymbol{I}}}$ as:

$$\begin{aligned}
I_{ij} &= \int dm\ (x_k x_k \delta_{ij} - x_i x_j)\omega_j \\
&= \int dm \begin{pmatrix} y^2 + z^2 & -xy & -xz \\ -xy & x^2 + z^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{pmatrix}
\end{aligned}$$

The kinetic energy may also be found in these terms:

$$\begin{aligned}
T &= \frac{1}{2}\int dm\ \dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} \\
&= \frac{1}{2}\int dm\ (\boldsymbol{\omega} \times \boldsymbol{r}) \cdot \dot{\boldsymbol{r}} \\
&= \frac{1}{2}\int dm\ \boldsymbol{\omega} \cdot (\boldsymbol{r} \times \dot{\boldsymbol{r}}) \\
&= \frac{1}{2}\boldsymbol{\omega} \cdot \int dm\ (\boldsymbol{r} \times \dot{\boldsymbol{r}}) \\
&= \frac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{J} = \frac{1}{2}\boldsymbol{\omega}\underline{\underline{\boldsymbol{I}}}\boldsymbol{\omega}
\end{aligned}$$

$\underline{\underline{I}}$ is symmetric, and so there are three mutually perpendicular "principal axes", with respect to which the tensor is diagonal; the diagonal components are termed the "principal moments of inertia" I_1, I_2 and I_3. If all three are equal, the body is described as a *spherical top*; if two are equal and the third different, it is a *symmetrical top*; if all three are different it is an *asymmetric top*. With respect to the principal axes, we see that:

$$\begin{aligned} I_1 + I_2 &= \int dm\ (y^2 + z^2 + x^2 + z^2) \\ &= I_3 + 2\int dm\ z^2 \geq I_3 \end{aligned}$$

and so no moment of inertia is greater than the sum of the other two; there is a sort of triangle inequality. The degenerate case, where $z = 0$ throughout the body, corresponds to a flat lamina, for which $I_1 + I_2 = I_3$: the *perpendicular axes theorem*.

Where it is inconvenient or incomplete to analyse rotation about the centre of mass, one can use the *parallel axis theorem* to find a principal moment about a new axis parallel to the principal axes going through the centre of mass. Consider $I_3 = \int dm\ (x^2 + y^2) = \int dm\ r^2$ with the axes through the centre of mass. If the axes are displaced by a distance a, then I_3' with respect to these parallel axes becomes:

$$\begin{aligned} I_3' &= \int dm\ (r+a)^2 = \int dm\ r^2 + 2a\underbrace{\int dm\ r}_{0} + a^2\underbrace{\int dm}_{M} \\ &= I_3 + Ma^2 \end{aligned}$$

where M is the mass of the whole body.

The surface $T(\boldsymbol{\omega})$ in $\boldsymbol{\omega}$-space is an ellipsoid. With respect to its principal axes, the ellipsoid is:

$$T = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2$$

We see that the semi-axes of the ellipsoids are inversely proportional to $\sqrt{I_i}$, so the longest axis corresponds to the smallest principal moment. Further, the normal vector to the ellipsoid is:

$$\boldsymbol{\nabla}_{\boldsymbol{\omega}} T = I_1\omega_1 + I_2\omega_2 + I_3\omega_3 = \boldsymbol{J}$$

And therefore $\boldsymbol{J} \parallel \boldsymbol{\omega}$ only if $\boldsymbol{\omega}$ is aligned along one of the principal axes of the body, otherwise the two are counter-intuitively in different directions.

2.3 Precession

Even when there are no forces on a rigid body, it can still rotate chaotically. If $\boldsymbol{J} \parallel \boldsymbol{\omega}$, the system is easy to describe, but otherwise $\boldsymbol{\omega}$ varies (both in the body frame and the space frame!); for asymmetric tops, $\boldsymbol{\omega}$ even varies in magnitude! As such, intuition is of limited use, and one should mostly rely on the mathematics.

Almost throughout, we will only consider symmetric tops, i.e. $I_1 = I_2 \neq I_3$.

Applying the operator equation at the start of this chapter to $\boldsymbol{J}$, we find *Euler's equations*, cause he doesn't have enough already:

$$\left[\frac{d\boldsymbol{J}}{dt}\right]_{S_0} = \left[\frac{d\boldsymbol{J}}{dt}\right]_S + \boldsymbol{\omega} \times \boldsymbol{J}$$
$$\Rightarrow \boldsymbol{G} = \left[\frac{d\boldsymbol{J}}{dt}\right]_S + \boldsymbol{\omega} \times \boldsymbol{J}$$

It is particularly crucial at this point to note that these (and following) equations apply only at a single instant of time, as a moment later the axes will have changed. $\boldsymbol{J}$ is the same in both frames, but $d\boldsymbol{J}/dt$ is not because one set of axes is moving. It is chosen that the axes are set along the principal axes of the body at a given instant, in which $\boldsymbol{\omega}$ is instantaneously described as $(\omega_1, \omega_2, \omega_3)$, with $\boldsymbol{J} = (I_1\omega_1, I_2\omega_2, I_3\omega_3)$. The above equation can then be written in component form as the following three coupled non-linear differential equations:

$$G_1 = I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3$$
$$G_2 = I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1$$
$$G_3 = I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2$$

Not that it's a massive drag but only one of these equations need be memorised; the other two follow by cyclic permutation. If $I_1 = I_2$ these simplify:

$$G_1 = I_1\dot{\omega}_1 + (I_3 - I_1)\omega_2\omega_3$$
$$G_2 = I_1\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1$$
$$G_3 = I_3\dot{\omega}_3$$

2.3.1 Free Precession

If there are no forces on a body, we have $\boldsymbol{G} = \mathbf{0}$, and Euler's equations become:

$$
\begin{aligned}
0 &= I_1\dot{\omega_1} + (I_3 - I_2)\omega_2\omega_3 \\
0 &= I_2\dot{\omega_2} + (I_1 - I_3)\omega_3\omega_1 \\
0 &= I_3\dot{\omega_3} + (I_2 - I_1)\omega_1\omega_2
\end{aligned}
$$

Dividing the first equation by the second:

$$
\begin{aligned}
\frac{I_1\dot{\omega_1}}{I_2\dot{\omega_2}} &= \frac{(I_2 - I_3)\omega_2}{(I_3 - I_1)\omega_1} \\
I_1(I_3 - I_1)\omega_1\dot{\omega_1} &= I_2(I_2 - I_3)\omega_2\dot{\omega_2} \\
I_1(I_3 - I_1)\omega_1^2 &= I_2(I_2 - I_3)\omega_2^2 + \text{const.} \\
\Rightarrow \frac{\omega_1^2}{I_2(I_2 - I_3)} + \frac{\omega_2^2}{I_1(I_1 - I_3)} &= \text{const.}
\end{aligned}
$$

This is an ellipsoid if I_3 is either the largest or smallest of the I_i, but a hyperbola if I_3 is the intermediate axis. Thus rotation is only stable if it is about the largest or smallest axes.

For a symmetric top, Euler's equations become:

$$
\begin{aligned}
0 &= I_1\dot{\omega_1} + (I_3 - I_1)\omega_2\omega_3 \\
0 &= I_1\dot{\omega_2} + (I_1 - I_3)\omega_3\omega_1 \\
0 &= I_3\dot{\omega_3}
\end{aligned}
$$

The final equation shows that ω_3 is a constant. Furthermore, the other two simplify by defining the *body frequency* $\Omega_b = \frac{I_1 - I_3}{I_1}\omega_3$:

$$
\begin{aligned}
0 &= \dot{\omega_1} - \Omega_b\omega_2 \\
0 &= \dot{\omega_2} + \Omega_b\omega_1
\end{aligned}
$$

These equations simplify on substitution to simple harmonic equations. They are slightly more restricted though, and the general solution is, for example:

$$
\begin{aligned}
\omega_1 &= A\sin(\Omega_b + \phi_0) \\
\omega_2 &= A\cos(\Omega_b + \phi_0)
\end{aligned}
$$

When combined with a constant ω_3 then, we see that $\boldsymbol{\omega}$ precesses around the 3-axis of the body, tracing out the *body cone*. Its tip (always on the surface of the inertia ellipsoid) traces out a circle known as the *polhode*. Throughout, $\boldsymbol{J}$ also precesses about the 3-axis at the same frequency Ω_b.

In the space frame, however, $\boldsymbol{J}$ does not change as $\boldsymbol{G} = \boldsymbol{0}$. $\boldsymbol{\omega}$ then precesses about the fixed $\boldsymbol{J}$, at the *space frequency* Ω_s, which turns out to be a bit fiddly to calculate but has a simple form. Consider the state of the system when $\omega_1 = 0$: $\boldsymbol{\omega} = (0, \omega_2, \omega_3)$ and $\boldsymbol{J} = (0, I_1\omega_2, I_3\omega_3)$. The component of $\boldsymbol{\omega}$ perpendicular (ω_P) to $\boldsymbol{J}$ is given by:

$$\omega_P = |\boldsymbol{\omega}| \sin\theta_s = \frac{|\boldsymbol{J} \times \boldsymbol{\omega}|}{|\boldsymbol{J}|} = \frac{|I_1 - I_3|\omega_2\omega_3}{J}$$

Now in the same way that for a circular path $|v| = r\omega$, one might realise that $|\dot{\omega}_1| = \omega_P \Omega_s$ where $\omega_1 = 0$. Using Euler's equation for $\dot{\omega}_1$, we therefore have:

$$\Omega_s = \frac{|\dot{\omega}_1|}{\omega_P} = \frac{|I_1 - I_3|\omega_2\omega_3}{I_1} \frac{J}{|I_1 - I_3|\omega_2\omega_3} = \frac{J}{I_1}$$

If the body is not perfectly rigid, but there are still no external forces, the ellipsoid essentially shrinks. Although in the body frame $\boldsymbol{J}$ is moving, it is not in the space frame, and further the magnitude is the constant in both frames. As the $\boldsymbol{J}$ vector must remain on the ellipsoid, the smallest that the ellipsoid can be is if the shortest semi-axis is aligned with $\boldsymbol{J}$. In other words, $\boldsymbol{J}$ gradually becomes aligned with the major axis. It is easily seen that this corresponds to the smallest kinetic energy for a given $|\boldsymbol{J}|$. This is the *major axis theorem*.

2.3.2 Poinsot Construction

The Poinsot construction illustrates some of the above quantities visually, using the inertia ellipsoid. Now as $\boldsymbol{J}$ and $T = \frac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{J}$ are conserved, the component of $\boldsymbol{\omega}$ in the $\boldsymbol{J}$ direction is constant. If one leaves the $\boldsymbol{J}$ direction pointed downwards, $\boldsymbol{\omega}$ remains in a horizontal plane, known as the *invariable plane*. Further, as $\boldsymbol{J}$ is perpendicular to the ellipsoid, the ellipsoid is tangential to the plane. Now because the instantaneous motion is a rotation of the ellipsoid about $\boldsymbol{\omega}$, it *rolls*, without slipping, on the invariable plane. The tip of the $\boldsymbol{\omega}$ vector traces out a circle known as the *herpolhode* on the invariable plane, and the vector itself traces out the *body cone*, which rolls over the space cone as the body rotates.

2.3.3 Euler Angles

Apparently nobody else bothered to study rotation. Another way of looking at a symmetric top is to use a set of angles known as the *Euler angles*. The first two, θ and ϕ, are simply the spherical polar angles for the 3-axis of the body, with respect to some convenient (and instantaneously stationary) axes $\{\hat{\boldsymbol{e}}_x, \hat{\boldsymbol{e}}_y, \hat{\boldsymbol{e}}_z\}$. The third, χ is the angle of rotation about the 3-axis – a component of it is in the same direction as ϕ. Supposing that the 1-axis of the body is instantaneously along the $\hat{\boldsymbol{e}}_\phi$ direction, the total angular velocity of the body is given by:

$$\boldsymbol{\omega} = \dot{\phi}\hat{\boldsymbol{e}}_z + \dot{\theta}\hat{\boldsymbol{e}}_1 + \dot{\chi}\hat{\boldsymbol{e}}_3$$

or, since $\hat{\boldsymbol{e}}_z = \hat{\boldsymbol{e}}_3 \cos\theta + \hat{\boldsymbol{e}}_2 \sin\theta$,

$$\boldsymbol{\omega} = \left(\dot{\theta}, \dot{\phi}\sin\theta, (\dot{\chi} + \dot{\phi}\cos\theta)\right)$$

and, remembering that the top is symmetric, $\boldsymbol{J}$ is then given by:

$$\boldsymbol{J} = \left(I_1\dot{\theta}, I_1\dot{\phi}\sin\theta, I_3(\dot{\chi} + \dot{\phi}\cos\theta)\right)$$

A first use of this is to rederive the space and body frequencies, by taking the 3-axis to be along $\boldsymbol{J}$. In this way, the body frequency is simply $\Omega_b = \dot{\chi}$ and the space frequency is $\Omega_s = \dot{\phi}$. Thus considering the 2-axis we have:

$$J\sin\theta = I_1\Omega_s \sin\theta \;\Rightarrow\; \Omega_s = \frac{J}{I_1}$$

Considering the 3-axis:

$$\omega_3 = \Omega_b + \frac{J\cos\theta}{I_1} = \Omega_b + \frac{I_3}{I_1}\omega_3 \;\Rightarrow\; \Omega_b = \frac{I_1 - I_3}{I_1}\omega_3$$

both as before.

2.4 Forced Precession & Gyroscopes

For a gyroscope and similar systems, it is often the case that the forces produce a torque in the $\hat{\boldsymbol{e}}_1$ direction. One could use Euler's equations to analyse the system, though the Euler angle approach is often easier to visualise in the space frame.

As $\boldsymbol{G} = G_1\hat{\boldsymbol{e}}_1$, the quantities J_2 and J_3 are instantaneously conserved, as are any linear combination. Therefore J_3 and $J_z = J_2\sin\theta + J_3\cos\theta$ are both conserved quantities, and they depend on $\dot{\chi}$ and $\dot{\phi}$, as well as θ. As such, $\dot{\chi}$ and $\dot{\phi}$ can be rewritten in terms of constants J_3, J_z and θ. The total kinetic energy T of the body can therefore be written in terms of $\dot{\theta}$ and θ. Often, including for a gyroscope, the potential energy U can also be written in terms of θ, and so the total energy E is simply a function of $\dot{\theta}$ and θ – the problem has been reduced to 1 dimension. It is found that, for a gyroscope whose centre of mass is h away from the pivot:

$$E = \frac{1}{2} I_1 \dot{\theta}^2 + \underbrace{\frac{(J_z - J_3 \cos\theta)^2}{2I_1 \sin^2\theta} + mgh\cos\theta + \frac{J_3^2}{2I_3}}_{U_{\text{eff}}(\theta)}$$

For steady precession, U_{eff} takes its minimum value and θ (and by extension $\dot{\phi}$ and $\dot{\chi}$) is constant. It turns out that this gives the following quadratic:

$$I_1 \cos\theta \dot{\phi}^2 - J_3 \dot{\phi} + mgh = 0$$

with the solutions:

$$\dot{\phi} = \frac{J_3 \pm \sqrt{J_3^2 - 4I_1 mgh\cos\theta}}{2I_1 \cos\theta}$$

This illustrates several important ideas. Firstly, J_3 must be larger than a certain value for the stationary precession to be possible – this is intuitively because the gyroscope must be spinning at a certain rate in order to maintain its motion. The *gyroscopic limit* describes the situation where J_3 is very large. In this case, the two possible precession frequencies are:

- "Slow Precession" (usually seen) – neglect $\dot{\phi}^2$ term in quadratic, giving $\dot{\phi} \approx mgh/J_3$
- "Fast Precession" – neglect mgh, giving $\dot{\phi} = J_3/I_1 \cos\theta$, which is in fact equal to Ω_s

If E is greater than the minimum of U_{eff}, θ oscillates and the gyroscope *nutates*. If E is only slightly greater than the minimum, this nutation approaches SHM, but for larger energies the nutation is very complicated.

3 Normal Modes

Suppose that the general state of a system is specified by N generalised coordinates, denoted $\{q_i\}$ or $\boldsymbol{q}$; more specifically, suppose $T = T(\boldsymbol{q})$ and $U = U(\boldsymbol{q})$. Without loss of generality, assume that $U(\mathbf{0}) = 0$ and $\nabla U|_{\boldsymbol{q}=\mathbf{0}} = \mathbf{0}$, i.e. there is an equilibrium at $\boldsymbol{q} = \mathbf{0}$ and the arbitrary constant is defined to be 0 here. For small displacements about equilibrium, we then have:

$$U(\boldsymbol{q}) \approx \underbrace{U(\mathbf{0})}_{=0} + \sum_i \underbrace{\left.\frac{\partial U}{\partial q_i}\right|_{\boldsymbol{q}=\mathbf{0}}}_{=0} q_i + \frac{1}{2}\sum_{i,j} \left.\frac{\partial^2 U}{\partial q_i \partial q_j}\right|_{\boldsymbol{q}=\mathbf{0}} q_i q_j$$

$$= \frac{1}{2}\sum_{i,j} \left.\frac{\partial^2 U}{\partial q_i \partial q_j}\right|_{\boldsymbol{q}=\mathbf{0}} q_i q_j \equiv \frac{1}{2}\boldsymbol{q}^\intercal \underline{\underline{\boldsymbol{K}}}\boldsymbol{q}$$

Where $\underline{\underline{\boldsymbol{K}}}_{ij} = \frac{\partial^2 U}{\partial q_i \partial q_j}|_{\boldsymbol{q}=\mathbf{0}}$. The kinetic energy is calculated similarly, and we have:

$$T(\boldsymbol{q}) = \frac{1}{2}\dot{\boldsymbol{q}}^\intercal \underline{\underline{\boldsymbol{M}}}\dot{\boldsymbol{q}}$$

where the dependence is now on $\dot{\boldsymbol{q}}$; $\underline{\underline{\boldsymbol{M}}}$ can also be chosen to be symmetric. There may be some residual dependence on $\boldsymbol{q}$, but to avoid double derivative jeopardy, the $\boldsymbol{q}$ dependence should simply be evaluated at $\boldsymbol{q} = \mathbf{0}$. The total energy is therefore:

$$E = \frac{1}{2}\boldsymbol{q}^\intercal \underline{\underline{\boldsymbol{K}}}\boldsymbol{q} + \frac{1}{2}\dot{\boldsymbol{q}}^\intercal \underline{\underline{\boldsymbol{M}}}\dot{\boldsymbol{q}}$$

Taking the time derivative with the knowledge that $\dot{E} = 0$,

$$0 = \frac{1}{2}\dot{\boldsymbol{q}}^\intercal \underline{\underline{\boldsymbol{K}}}\boldsymbol{q} + \frac{1}{2}\boldsymbol{q}^\intercal \underline{\underline{\boldsymbol{K}}}\dot{\boldsymbol{q}} + \frac{1}{2}\ddot{\boldsymbol{q}}^\intercal \underline{\underline{\boldsymbol{M}}}\dot{\boldsymbol{q}} + \frac{1}{2}\dot{\boldsymbol{q}}^\intercal \underline{\underline{\boldsymbol{M}}}\ddot{\boldsymbol{q}}$$

but because each matrix is symmetric and the quadratic forms are scalars, the colour-coded terms are equal, and we have:

$$0 = \dot{\boldsymbol{q}}^\intercal \underline{\underline{\boldsymbol{K}}}\boldsymbol{q} + \dot{\boldsymbol{q}}^\intercal \underline{\underline{\boldsymbol{M}}}\ddot{\boldsymbol{q}}$$
$$= \dot{\boldsymbol{q}}^\intercal \left(\underline{\underline{\boldsymbol{K}}}\boldsymbol{q} + \underline{\underline{\boldsymbol{M}}}\ddot{\boldsymbol{q}}\right)$$

and since $\dot{\boldsymbol{q}}^\intercal$ is not in general $\mathbf{0}$, we must have:

$$\underline{\underline{\boldsymbol{K}}}\boldsymbol{q} + \underline{\underline{\boldsymbol{M}}}\ddot{\boldsymbol{q}} = \mathbf{0} \quad (3.1)$$

We now restrict the solutions to oscillatory solutions, positing that:

$$\boldsymbol{q} = \boldsymbol{q}_0 e^{i\omega t}$$

which, substituting into 3.1 and assuming $\boldsymbol{q}_0 \neq \mathbf{0}$, gives:

$$\left(\underline{\underline{\boldsymbol{K}}} - \omega^2 \underline{\underline{\boldsymbol{M}}}\right)\boldsymbol{q}_0 = \mathbf{0} \Rightarrow \left|\underline{\underline{\boldsymbol{K}}} - \omega^2 \underline{\underline{\boldsymbol{M}}}\right| = 0$$

an eigenvalue problem that can be solved for ω and then $\boldsymbol{q}_0$, which may not be real. If any eigenvalues ω^2 are negative, this corresponds to modes which are not in fact stable, and the initial presumption of stable oscillatory solutions was incorrect. It is quite possible for eigenvalues to be 0, corresponds to some direction in $\boldsymbol{q}$-space in which $\underline{\underline{\boldsymbol{K}}}$ is zero; returning to (3.1) this clearly has translational solutions of the form: $\boldsymbol{q} = \boldsymbol{q}_1 t + \boldsymbol{q}_0$.

Note that the eigenvectors (the $\boldsymbol{q}_0$) need not be orthogonal.

4 Elasticity

4.1 Young's Modulus and Poisson's Ratio

When a thin wire of length l and cross-sectional area A has a small force F applied to it, it will extend in the direction of the force by a distance δl. The Young's Modulus E is defined by:

$$E = \frac{Fl}{A\delta l} \Rightarrow \frac{F}{A} = E\frac{\delta l}{l}$$

The quantity F/A, force per unit area, is referred to as the *stress* τ, and $\delta l/l$, fractional extension, is referred to as the *strain* e. So $Ee = \tau$.

When a block of material is stretched by δl in one direction, it compresses in the other two directions by a distance $\sigma\delta l$, where σ is the *Poisson's ratio*. For a linear isotropic material, E and σ completely determine its elastic behaviour. In particular, with respect to some orthogonal axes, we have the relations:

$$Ee_1 = \tau_1 - \sigma(\tau_2 + \tau_3)$$

and the cyclic permutations thereof.

4.2 Tensor Generalisations

One can generalise this to include *shear* stresses where a force is *across* a face, by reformulating the stress as a tensor $\underline{\underline{\boldsymbol{\tau}}}$. We then have, for a surface $d\boldsymbol{S}$:

$$d\boldsymbol{F} = \underline{\underline{\boldsymbol{\tau}}} \cdot d\boldsymbol{S}$$

The $\underline{\underline{\boldsymbol{\tau}}}$ tensor is symmetic in equilibrium, as if, say, $\tau_{xy} \neq \tau_{yx}$, then this would create a net couple on the body. As such, $\underline{\underline{\boldsymbol{\tau}}}$ is diagonalisable and there exist some axes with respect to which all stresses are normal rather than shear. We can then introduce a vector field $\boldsymbol{u}$ known as the displacement; at a point $\boldsymbol{r}$, $\boldsymbol{u}(\boldsymbol{r})$ is the displacement that point moves through when the stress is applied. Another tensor, $\underline{\underline{e}}$, is used to define the strain. The diagonal terms account for normal strains; the off-diagonals account for shear strains. It is clear that, for instance:

$$e_{xx} = \frac{\partial u_x}{\partial x},$$

and so

$$Ee_{xx} = \tau_{xx} - \sigma\left(\tau_{yy} + \tau_{zz}\right),$$

assuming there are no shear stresses. The *shear strain* is defined by, for instance:

$$e_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

Infinitesimally, this corresponds to the total angle between an original unstressed cube to the stressed diamond that shear stresses cause. Further, we see that $\underline{\underline{e}}$ is symmetric; for an isotropic material (with which we will solely be concerned), it is diagonal with respect to the same axes as $\underline{\underline{\tau}}$. We can write the general definition of the strain tensor components as:

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

4.3 Bulk and Shear Moduli

Suppose a material is under an isotropic pressure, so that $\tau_1 = \tau_2 = \tau_3 = -P$. Thus from the above equations, $e_1 = e_2 = e_3 = -P(1-2\sigma)/E$. The fractional change in volume is then

$$\begin{aligned}\frac{\delta V}{V} &= (1+e_1)(1+e_2)(1+e_3) - 1 \\ &\approx e_1 + e_2 + e_3 \\ &= -\frac{3P(1-2\sigma)}{E} \quad \Rightarrow P = -\frac{E}{3(1-2\sigma)}\frac{\delta V}{V}\end{aligned}$$

The bulk modulus B is defined by:

$$P = -B\frac{\delta V}{V} \quad \Rightarrow \quad B = \frac{E}{3(1-2\sigma)}$$

B has to be positive, as otherwise work done by the pressure would lead to an extension in the opposite direction to the forces. Therefore $\sigma < 0.5$; usually $\sigma \approx 0.3$.

The shear modulus G is defined as the ratio between the shear stress and the shear strain, i.e.:

$$\tau_{xy} = G\gamma_{xy}$$

where γ_{xy} is the *shear angle*, equal to $2e_{xy}$. It turns out then that:

$$G = \frac{E}{2(1+\sigma)}$$

Similar considerations as for B constrain that $\sigma > -1$, not that it's usually very close.

4.4 Beams

When a beam experiences a bending moment (e.g. due to its weight, or to a weight placed on it), it responds by distorting into an arc of radius R. There will be a plane through which the beam is not extended at all: the *neutral axis*; on one side of this axis (usually below) the beam is compressed; on the other the beam is extended. At a distance y above (say) the neutral axis, the lateral extension is y/R, so the total bending moment is:

$$B = \int y\tau \; dA = \int yE\frac{y}{R} \; dA = \frac{E}{R}\int y^2 \; dA = \frac{EI}{R}$$

where I is the moment of area in the y-direction. Now $1/R$ is approximately equal to y'', so we have the important formula:

$$B = EIy''$$

B is often found by physical considerations for a particular beam, as a function of x, which on integration (setting $y(0) = 0$ arbitrarily, and considering the boundary conditions below) gives the shape of the graph. A similar equation arises if there is a distributed load $W(x)$ per unit length along the beam:

$$W = EIy''''$$

Oh and don't worry about negative signs, it should be obvious which way round is the right way. There are three common boundary conditions one should consider before finding $B(x)$:

- **Free end** — no force or couple provided by end $\Rightarrow y''(0) = 0$
- **Hinge** — force, but no couple, provided by end $\Rightarrow y''(0) = 0, y'(0) \neq 0$
- **Cantilever** — force and couple provided by end $\Rightarrow y'(0)$ given

4.5 Elastic Dynamics

4.5.1 Bulk Media

The F_x on an infinitesimal volume V is clearly given by:

$$F_x = V\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}\right)$$

$$\Rightarrow F_i = V\frac{\partial \tau_{ij}}{\partial x_j}$$

$$\Rightarrow \rho\frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \tau_{ij}}{\partial x_j} \qquad \text{or } \rho\ddot{\mathbf{u}} = \boldsymbol{\nabla}\underline{\underline{\boldsymbol{\tau}}}$$

Considerations of the strain-stress relations give that:

$$\underline{\underline{\tau}} = \left(B - \frac{2}{3}G\right) \mathrm{Tr}(\underline{\underline{e}})\underline{\underline{\boldsymbol{I}}} + 2G\underline{\underline{e}}$$

where $\underline{\underline{\boldsymbol{I}}}$ is the identity tensor. Anyway this eventually leads to:

$$\rho\ddot{\boldsymbol{u}} = \left(B + \frac{1}{3}G\right) \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) + G\boldsymbol{\nabla}^2\boldsymbol{u}$$

This actually has normal mode solutions. For a wave travelling in the x-direction, we might have:

$$\boldsymbol{u} = (X_0, Y_0, Z_0)e^{i(\omega t - kx)}$$

where (X_0, Y_0, Z_0) is the arbitrary vector amplitude of the wave, and for which:

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = -ikX_0e^{i(\omega t - kx)} \qquad \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) = -k^2(X_0, 0, 0)e^{i(\omega t - kx)}$$
$$\boldsymbol{\nabla}^2\boldsymbol{u} = -k^2(X_0, Y_0, Z_0)e^{i(\omega t - kx)} \qquad \ddot{\boldsymbol{u}} = -\omega^2(X_0, Y_0, Z_0)e^{i(\omega t - kx)}$$

We therefore have:

$$\rho\omega^2(X_0, Y_0, Z_0) = k^2 \left[\left(B + \frac{1}{3}G\right)(X_0, 0, 0) + G(X_0, Y_0, Z_0)\right]$$

Considering the compressional (or "P") part of the wave (X_0), we have:

$$\frac{\omega^2}{k^2} = \frac{B + \frac{4}{3}G}{\rho},$$

whereas for either of the shearing components ("S"), we have:

$$\frac{\omega^2}{k^2} = \frac{G}{\rho}$$

We see that:

- The speed of compressional waves is always greater than that of shear waves
- Liquids and gases cannot support shear waves

4.5.2 Surfaces

Boundaries can either be fixed or free: fixed boundaries have no normal displacement at the boundary; free boundaries have no normal stress at the boundary.

5 Fluids

5.1 Archimedes' Principle

This principle states that the upthrust force experienced by a submerged body is equal to the weight of the fluid displaced, $\rho g V$ where ρ is the volume of the fluid and V the volume of the body. This force acts at the geometric centre of the object, that is, the centre of mass of the displaced fluid, rather than the centre of mass of the object; the net result is often a couple if the body does not have a uniform density.

5.2 Euler's Equation

Yep, him again. Considering a spatially-varying pressure $P(\boldsymbol{r})$, it becomes clear that the force per unit volume on a volume element is $-\boldsymbol{\nabla} P$. Including gravity, we obtain Euler's equation, which has not assumed incompressibility:

$$\rho \frac{D\boldsymbol{v}}{Dt} = -\boldsymbol{\nabla} P + \rho \boldsymbol{g}$$

The derivative on the LHS is not equal to $\partial \boldsymbol{v}/\partial t$, as the fluid element under consideration is also moving to a new point with a different velocity; the velocity field is predefined at different points. This is known as the *convective derivative.*

The change in $\boldsymbol{v}$ as one moves along with the fluid is given by:

$$\begin{aligned} d\boldsymbol{v}|_{\text{with fluid}} &= \frac{\partial \boldsymbol{v}}{\partial t} dt + d\boldsymbol{r} \cdot \boldsymbol{\nabla} \boldsymbol{v} \\ &= dt \left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} \right) \\ \Rightarrow \frac{D\boldsymbol{v}}{Dt} &= \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} \end{aligned}$$

Phrased better, $D\boldsymbol{v}/Dt$ is not the change in velocity of the fluid present at position $\boldsymbol{r}$, but rather the change in velocity of the fluid element instantaneously at that position; it will soon move to a new one. There is more generally

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla}$$

5.3 Approximations to Fluids

One approximation that is often made is that of *incompressibility*, which implies that ρ is a constant in space and time. Conservation of mass gives:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0$$

and therefore, for incompressible fluids, $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$.

Another common approximation is that of *irrotational* flow, in other words $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{v} = \boldsymbol{0}$. This is often the case when one is in the bulk of the fluid, as it is only usually boundaries which lead to vorticity. $\boldsymbol{v}$ can therefore be described as the gradient of a vector field $\boldsymbol{v} \equiv \boldsymbol{\nabla}\Phi$ (note the sign convention – $\boldsymbol{v}$ points "uphill"!).

The combination of irrotational and incompressible flows gives *potential flow*, as we then have $\boldsymbol{\nabla}^2\Phi = 0$, which is Laplace's equation. Often one uses Laplace's equation to find Φ and $\boldsymbol{v}$, and then finds the pressure using Bernoulli's Equation (below) If the pressure is found to go *negative* anywhere, then *cavitation* occurs, with bubbles forming and such.

If there is a source or sink present, we have $\boldsymbol{\nabla}^2\Phi = Q\delta(\boldsymbol{r})$, where Q is the mass flux in kg s^{-1} giving $\Phi = -Q/4\pi r$ and $v = Q/4\pi r^2$, so that mass is conserved as expected.

There exist *vortex* solutions of Laplace's equation, where Φ takes the form:

$$\Phi = \frac{\kappa\theta}{2\pi}$$

When one adds together the steady flow and vortex flow solutions, one obtains a flow with a net force in one direction, given by $F = \rho \boldsymbol{v}_0 \times \boldsymbol{\kappa}$ and known as the *Magnus force.*

Another approximation is that of *inviscid* fluids (AKA "dry" fluids). These are those for which the viscosity (see later) $\eta = 0$. A fluid which is both incompressible and irrotational (note – this is really a property of the fluid rather than the situation in question) is said to be an *ideal* fluid.

5.4 Bernoulli's Equation

Steady flows, those where $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{r})$ rather than $\boldsymbol{v}(\boldsymbol{r}, t)$, are easy to analyse by conservation of energy. Consider a pipe, where the fluid is rushing into an area A_1 at a speed v_1, a density ρ_1, pressure P_1, gravitational potential ϕ_1. The energy flow rate going in is therefore:

$$-\dot{E}_1 = A_1 v_1 \left(\frac{1}{2}\rho_1 v_1^2 + \rho\phi_1 \right) + A_1 P_1 v_1$$

where the last term (somewhat is the rate at which work is being done on the fluid ($= F_1 v_1$). Furthermore, mass conservation implies:

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2$$

So when rearranging the previous equation to:

$$-\dot{E}_1 = A_1 v_1 \rho_1 \left(\frac{1}{2} v_1^2 + \phi_1 + \frac{P_1}{\rho_1} \right)$$

We see that this must be equal to the energy flow rate going in, $\dot{E}_2$, which has the same symbols but with different subscripts. Since the prefactor term will be the same by mass conservation, the quantity

$$\frac{1}{2} v^2 + \phi + \frac{P}{\rho}$$

must also be the same on both sides. This quantity is therefore conserved. For incompressible fluids, ρ is also the same on both ends, so we can express this conserved quantity more conveniently:

$$\frac{1}{2} \rho v^2 + \rho \phi + P$$

which is essentially the energy per volume which is being pumped into the pipe. That this quantity is conserved is an expression of Bernoulli's "Equation".

5.5 Viscosity

In viscous fluids, there are non-negligible frictional forces between adjacent layers of the fluid in relative motion. A fluid contained between two plates, where there is a shear force on, say, the top plate, will induce the layer of fluid directly below the top plate to move at a speed equal to that of the plate. The layer directly above the *bottom*, stationary plate, will not be moving due to friction with this plate. The shearing force on the top plate, τ_{xy}, thus sets up a velocity gradient $\partial v_x / \partial y$ within the fluid; the following equation defines the viscosity:

$$\tau_{xy} = \eta \frac{\partial v_x}{\partial y}$$

Viscosity is thus a stress per velocity gradient, with units of Pa s.

It can be shown that this adds an extra term in Euler's equation of motion for a fluid, namely:

$$\rho \frac{D\boldsymbol{v}}{Dt} = -\boldsymbol{\nabla} P + \rho \boldsymbol{g} + \eta \left(\boldsymbol{\nabla}^2 \boldsymbol{v} + \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \boldsymbol{v}) \right)$$

or, for an incompressible fluid:

$$\rho \frac{D\boldsymbol{v}}{Dt} = -\boldsymbol{\nabla} P + \rho \boldsymbol{g} + \eta \boldsymbol{\nabla}^2 \boldsymbol{v}$$

and for steady flows of incompressible fluids:

$$0 = -\boldsymbol{\nabla} P + \rho \boldsymbol{g} + \eta \boldsymbol{\nabla}^2 \boldsymbol{v}$$

Often this is most easily solved by resolving into the x and y directions separately, so that some components (e.g. $\boldsymbol{g}$) only appear in one of the equations.

The Reynolds number is defined by:

$$\mathrm{Re} \equiv \frac{\rho v L}{\eta}$$

where L is the characteristic length scale of the setup (for example, the diameter of a sphere under consideration). If Re is less than about 10^5, the flow is *laminar*, so streamlines are unbroken and viscous stresses dominate the physics. If Re is *higher* than about 10^5, the flow can easily become *turbulent* and chaotic, as the fluid becomes dominated by *inertial* stresses.