Electromagnetic Fields

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Lockdown 2.0

0 Maxwell's Equations

$$\nabla \cdot \boldsymbol{D} = \rho_f \tag{1}$$
$$\nabla \cdot \boldsymbol{B} = 0 \tag{2}$$
$$\nabla \times \boldsymbol{E} = -\dot{\boldsymbol{B}} \tag{3}$$
$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_f + \dot{\boldsymbol{D}} \tag{4}$$

0.1 Other relevant equations

Divergence & Stokes:

$$\int_V \nabla \cdot \boldsymbol{K} \, dV = \oint_{\partial V} \boldsymbol{K} \cdot d\boldsymbol{S}; \qquad \int_S \nabla \times \boldsymbol{K} \cdot d\boldsymbol{S} = \oint_{\partial S} \boldsymbol{K} \cdot d\boldsymbol{r}$$

Continuity (follows from the divergence of (4)) and Conduction (Ohm's law):

$$\nabla \cdot \boldsymbol{J}_f = -\dot{\rho}_f; \qquad \boldsymbol{J} = \sigma \boldsymbol{E}$$

where σ is the material's conductivity (inverse of resistivity).

1 The Fields

1.1 $\boldsymbol{E}$

A charge q in an $\boldsymbol{E}$ field experiences a force:

$$\boldsymbol{F} = q\boldsymbol{E}$$

1.2 $\boldsymbol{D}$

The $\boldsymbol{D}$ field is defined as:

$$\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P}$$

$\boldsymbol{P}$ is the *polarisation density*, which accounts for dipoles created in a *dielectric* material due to an external $\boldsymbol{E}$ field. In the linear regime, $\boldsymbol{P} = \chi_e \epsilon_0 \boldsymbol{E}$, where χ_e is the *electric susceptibility* (in general a tensor). We thus have:

$$\begin{aligned} \boldsymbol{D} &= (1 + \chi_e)\epsilon_0 \boldsymbol{E} \\ &= \epsilon \epsilon_0 \boldsymbol{E} \end{aligned}$$

where $\epsilon = 1 + \chi_e$ is the *relative permittivity* of the medium.

1.3 $\boldsymbol{B}$

A moving charge in a $\boldsymbol{B}$ field experiences a force:

$$\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$$

Because of (2) and the fact that $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{K}) \equiv 0$, the $\boldsymbol{B}$ field can always be written as the curl of a different vector field, $\boldsymbol{A}$, so we have $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$. Because $\boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi \equiv 0$, the $\boldsymbol{A}$ field is actually only defined up to a gradient of an arbitrary scalar field, as well as an arbitrary constant, and $\boldsymbol{A}$ is often chosen so that $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$ ("Coulomb gauge")

1.4 $\boldsymbol{H}$

The $\boldsymbol{H}$ field is defined as:

$$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M}$$

$\boldsymbol{M}$ is the *magnetisation*, which accounts for dipoles created in a *magnetic* material. This may be due to an external $\boldsymbol{B}$ field, or the material may be a permanent magnet in which case a $\boldsymbol{M}$ field exists even with 0 external field. In the linear regime, $\boldsymbol{M} = \chi_m \boldsymbol{H}$, where χ_m is the *magnetic susceptibility* (also in general a tensor). We thus have:

$$\begin{aligned} \boldsymbol{H} &= \frac{\boldsymbol{B}}{\mu_0} - \chi_m \boldsymbol{H} \\ &= \frac{\boldsymbol{B}}{(1 + \chi_m)\mu_0} \\ &= \frac{\boldsymbol{B}}{\mu \mu_0} \end{aligned}$$

where $\mu = 1 + \chi_m$ is the *relative permeability* of the medium.

2 Static cases

2.1 Electrostatics

2.1.1 No dielectrics

Here (1) and (3) become:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} \tag{5}$$
$$\nabla \times \boldsymbol{E} = 0 \tag{6}$$

since $\boldsymbol{D}$ becomes equal to $\epsilon_0 \boldsymbol{E}$ and $\rho_f = \rho$ as there is no bound charge. (5)'s integral form (Gauss' Law) is useful if the system has some degree of symmetry:

$$\oint_{\partial V} \boldsymbol{E} \cdot d\boldsymbol{S} = \frac{1}{\epsilon_0} \int_V \rho \, dV$$

where the LHS is often called the *electric flux*, and often reduces to a scalar multiple of the constant $|\boldsymbol{E}|$ if there is some symmetry. From this the $\boldsymbol{E}$ around a uniform sheet charge, a uniform line charge, and a uniform point charge, are easily derived.
(6) means that we can write $\boldsymbol{E}$ as the gradient of a scalar field $-V$ (defined to within an arbitrary constant), so $\boldsymbol{E} = -\nabla V$ and $\nabla^2 V = -\frac{\rho_f}{\epsilon_0}$. Then, for an unbounded region we have, using the Green's function for the Laplacian with V going to zero at infinity:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathbb{R}^3} \frac{\rho(\boldsymbol{r'})}{|\boldsymbol{r} - \boldsymbol{r'}|} d^3\boldsymbol{r'}$$

Other boundary conditions often include the presence of conductors, which in electrostatics are *equipotentials*, having constant V (usually taken as 0) on their surface. These are generally harder to solve, and may be solved using the *method of images* or other methods, though if *any* satisfactory solution is found, then according to the uniqueness theorem it is the only solution (to within an arbitrary constant, which is present in V anyway). The $\boldsymbol{E}$ field can then be found from the V so derived.

2.1.2 Dielectrics

$$\nabla \cdot \boldsymbol{D} = \rho_f \tag{7}$$
$$\nabla \times \boldsymbol{E} = 0 \tag{8}$$

(5) is in fact generally true, as long as bound charge is included in this equation. Also, $\boldsymbol{\nabla} \cdot \boldsymbol{P} = -\rho_b$, and so $\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho - \rho_b = \rho_f$; this motivates the definition of $\boldsymbol{D}$, as it automatically "accounts for" the bound charge and we only have to worry about the free charge. If we insisted on only using $\boldsymbol{E}$, we would always have to consider the $\boldsymbol{P}$ and where the ρ_b was, but using $\boldsymbol{D}$ we are saved this effort.
At a boundary between two dielectric materials with a different ϵ, the normal component of $\boldsymbol{D}$ is continuous: there is no free charge contained within a pillbox enclosing a portion of the boundary, so there is no net flux of $\boldsymbol{D}$. That of $\boldsymbol{E}$ is discontinuous due to the bound surface charge created by the $\boldsymbol{P}$, however the *parallel* component of $\boldsymbol{E}$ *is* continuous, because of (8). From this the $\boldsymbol{E}$ around dielectric slabs and spheres abound can be derived, particularly if the dielectric is placed in an existing uniform field $\boldsymbol{E}_0$. It is often found that:

$$\boldsymbol{P} = \frac{\chi}{1 + n\chi} \epsilon_0 \boldsymbol{E}_0$$

where $0 < n < 1$. For a cylinder, $n = 1/2$; for a sphere, $n = 1/3$.

2.2 Magnetostatics

2.2.1 No magnetics

(2) and (4) become:

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \tag{9}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J} \tag{10}$$

since $\boldsymbol{H}$ becomes equal to $\boldsymbol{B}/\mu_0$ and $\boldsymbol{J}_f = \boldsymbol{J}$ as there is no bound current. (10)'s integral form (Ampère's law) is useful if there's symmetry:

$$\oint_{\partial S} \boldsymbol{B} \cdot d\boldsymbol{r} = \mu_0 \int_S \boldsymbol{J} \cdot d\boldsymbol{S}$$

The $\boldsymbol{B}$ around a wire and solenoid can then be calculated.
Because $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$, $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) - \boldsymbol{\nabla}^2 \boldsymbol{A}$ and $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$ (assuming Coulomb gauge), (10) gives:

$$\boldsymbol{\nabla}^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J}$$

Poisson's equation again, though in vector form. As before we can then write:

$$\boldsymbol{A} = \frac{\mu_0}{4\pi} \int_{\mathbb{R}^3} \frac{\boldsymbol{J}(\boldsymbol{r'})}{|\boldsymbol{r} - \boldsymbol{r'}|} \, d^3\boldsymbol{r'}$$

And taking the curl of this, we get (eventually!):

$$\boldsymbol{B} = \frac{\mu_0 I}{4\pi} \int_{\mathbb{R}^3} \frac{\boldsymbol{J}(\boldsymbol{r'}) \times (\boldsymbol{r} - \boldsymbol{r'})}{|\boldsymbol{r} - \boldsymbol{r'}|^3} \, d^3\boldsymbol{r'}$$

Alternatively, assuming current only travels along a thin wire, we can write $\boldsymbol{J}dV = JSd\boldsymbol{r} = Id\boldsymbol{r}$, and so the volume integral becomes a line integral as we have assumed a constant (small) area:

$$\boldsymbol{B} = \frac{\mu_0}{4\pi} \int_C \frac{d\boldsymbol{r'} \times (\boldsymbol{r} - \boldsymbol{r'})}{|\boldsymbol{r} - \boldsymbol{r'}|^3} \, d^3\boldsymbol{r'}$$

— the Biot-Savart Law.
In regions of 0 current, (10) becomes homogeneous and we can in fact write $\boldsymbol{B} = -\mu_0 \boldsymbol{\nabla}\phi_m$, the “magnetic scalar potential”. This has limited usability and is often multivalued in the regions *around* currents, though it can be useful if far away, especially if dealing with magnetic dipoles.

2.2.2 Magnetics

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$
$$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J}_f$$

The difference to the previous subsubsection is that now there is a $\boldsymbol{M}$ field (included in $\boldsymbol{H}$), and we have to only consider the free currents, since bound currents are now involved. (10) is also generally true, if bound currents are included. Also, $\boldsymbol{\nabla} \times \boldsymbol{M} = \boldsymbol{J}_b$, and so $\boldsymbol{\nabla} \times \boldsymbol{H} = \frac{\mu_0 \boldsymbol{J}}{\mu_0} - \boldsymbol{J}_b = \boldsymbol{J}_f$, motivating the definition of $\boldsymbol{H}$, which thus automatically accounts for currents caused by $\boldsymbol{M}$.
Similarly to dielectrics, at the boundary between two surfaces with different μ the normal component of $\boldsymbol{B}$ is continuous (because $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$) but that of $\boldsymbol{H}$ is discontinuous due to the surface currents created by $\boldsymbol{M}$; the parallel component of $\boldsymbol{H}$ is continuous because $\boldsymbol{J}_f = 0$. Similar problems to those posed in electrostatics can then be tackled, as well as electromagnets.

3 Some consequences

3.1 Dipoles

3.1.1 Electric dipoles

This is an electrostatic situation. We may use Gauss' Law to find that the $\boldsymbol{E}$ and V around a *single* charge q to be:

$$\boldsymbol{E} = \frac{q}{4\pi\epsilon_0 r^2}\boldsymbol{e}_r \Rightarrow V = \frac{q}{4\pi\epsilon_0 r}$$

where the arbitrary choice of $\lim_{r\to\infty} V = 0$ has been made. Now consider two equal and opposite charges, positioned at $\boldsymbol{r}$ and $\boldsymbol{r} + \delta\boldsymbol{r}$. The V field due to the two charges is then:

$$\begin{aligned} V &= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{|\boldsymbol{r} + \delta\boldsymbol{r}|}\right) \\ &= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r} - (\frac{1}{r} - \frac{\delta\boldsymbol{r}\cdot\boldsymbol{r}}{r^3} + \ldots)\right) \\ &\approx \frac{q\delta\boldsymbol{r}\cdot\boldsymbol{r}}{4\pi\epsilon_0 r^3} = \frac{\boldsymbol{p}\cdot\boldsymbol{r}}{4\pi\epsilon_0 r^3} \end{aligned}$$

where $\boldsymbol{p} = q\delta\boldsymbol{r}$; the *electric dipole moment*. It is usually convenient to define the z-axis to be parallel to $\boldsymbol{p}$, which gives:

$$V = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$

Unlike prior expressions for V, this one drops off as $1/r^2$, as the numerator now has a length dimension in p. From this we can then derive $\boldsymbol{E} = -\boldsymbol{\nabla}V$:

$$\boldsymbol{E} = \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\ \boldsymbol{e}_r + \sin\theta\ \boldsymbol{e}_\theta\right)$$

This is a classic field.
If the dipole is placed at a fixed point in an external $\boldsymbol{E}$ field, there will be a couple on the dipole. This couple can be easily shown to be:

$$\boldsymbol{G} = \boldsymbol{p} \times \boldsymbol{E}$$

Alternatively, if the dipole's *centre* is allowed to move, but the dipole's *orientation* is fixed (i.e. the opposite situation), the net force on the dipole can be shown to be:

$$\begin{aligned} \boldsymbol{F} &= (\boldsymbol{p}\cdot\boldsymbol{\nabla})\boldsymbol{E} \\ &= \boldsymbol{\nabla}(\boldsymbol{p}\cdot\boldsymbol{E}) \end{aligned}$$

since now $\boldsymbol{p}$ is now a constant. Interestingly, both the torque from before and the force just derived can be thought of as being the result of the dipole possessing a potential energy $U = -\boldsymbol{p} \cdot \boldsymbol{E}$ when in the field.

3.1.2 Magnetic dipoles

Unlike electric dipoles, magnetic dipoles create the simplest magnetic field possible (i.e. the simplest field with 0 divergence). It can be shown that when an infinitesimal loop of current I around a surface $d\boldsymbol{S}$ is placed inside a magnetic field $\boldsymbol{B}$, the loop experiences a torque:

$$d\boldsymbol{G} = I d\boldsymbol{S} \times \boldsymbol{B}$$

suggesting a definition of a *magnetic dipole moment* $\boldsymbol{m} = I\boldsymbol{S}$. This has essentially the same properties as the electric dipole, and when on its own it generates a $\boldsymbol{B}$ field :

$$\boldsymbol{B} = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\ \boldsymbol{e}_r + \sin\theta\ \boldsymbol{e}_\theta\right)$$

almost identically to the electric case.

3.2 Energy

In electrostatics, it is derived rather easily that:

$$U = \frac{1}{2}\int_{\mathbb{R}^3} \rho_f V d\tau$$

where U is the potential energy of the setup and V is the potential at a certain position. We can consider this potential energy, arising due to the placement of charges next to each other, as being "located" in the electric field, since there's not really anywhere else to place it. Anyway, we then have:

$$\begin{aligned} U &= \frac{1}{2}\int_{\mathbb{R}^3} (\boldsymbol{\nabla} \cdot \boldsymbol{D}) V d\tau \\ &= \frac{1}{2}\int_{\mathbb{R}^3} \left[\boldsymbol{\nabla} \cdot (\boldsymbol{D}V) - \boldsymbol{D} \cdot \boldsymbol{\nabla} V\right] d\tau \\ &= \frac{1}{2}\int_{\mathbb{R}^3} \boldsymbol{D} \cdot \boldsymbol{E}\ d\tau \end{aligned}$$

where we assumed that V goes to 0 at infinity, and used the divergence theorem with the identity $\boldsymbol{\nabla} \cdot (\boldsymbol{F}\phi) = \phi\boldsymbol{\nabla} \cdot \boldsymbol{F} + \boldsymbol{F} \cdot \boldsymbol{\nabla}\phi$, to derive this elegant expression. We can thus posit that the energy density stored in the electric field is then $\frac{1}{2}\boldsymbol{D}\cdot\boldsymbol{E}$.

A similar argument goes for magnetostatics. Consider a wire wound very tightly into a coil. If a current is passed through it, there will be a magnetic field through the middle, and if this current is time-varying then the change in the magnetic flux through the wire will generate an emf across the two ends of the wire, given by $\dot{\Phi}$. But Φ is proportional to the magnetic field, which (one can show by deriving the magnetic field within a solenoid) is proportional to the current. So we write $\Phi = LI$ (where L is the *self-inductance*) and the emf as $L\dot{I}$. The power then dissipated is:

$$
\begin{aligned}
P &= L\dot{I}I \\
&= \frac{1}{2}\frac{\partial}{\partial t}\left(LI^2\right) \\
\Rightarrow U &= \frac{1}{2}LI^2 \\
&= \frac{1}{2}\Phi I \\
&= \frac{1}{2}\int_{\text{wire}} I\boldsymbol{B}\cdot d\boldsymbol{S} \\
&= \frac{1}{2}\oint_{\text{wire}} I\boldsymbol{A}\cdot d\boldsymbol{r} \\
&= \frac{1}{2}\int_{\mathbb{R}^3} \boldsymbol{A}\cdot\boldsymbol{J}\ d\tau \\
&= \frac{1}{2}\int_{\mathbb{R}^3} \boldsymbol{A}\cdot(\boldsymbol{\nabla}\times\boldsymbol{H})\ d\tau \\
&= \frac{1}{2}\int_{\mathbb{R}^3} \left[\boldsymbol{H}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A}) - \boldsymbol{\nabla}\cdot(\boldsymbol{A}\times\boldsymbol{H})\right]\ d\tau \\
&= \frac{1}{2}\int_{\mathbb{R}^3} \boldsymbol{H}\cdot\boldsymbol{B}\ d\tau
\end{aligned}
$$

where we assumed that $\boldsymbol{A}$ goes to 0 at infinity, and used the divergence theorem with the identity $\boldsymbol{\nabla}\cdot(\boldsymbol{F}\times\boldsymbol{G}) = \boldsymbol{G}\cdot(\boldsymbol{\nabla}\times\boldsymbol{F}) - \boldsymbol{F}\cdot(\boldsymbol{\nabla}\times\boldsymbol{G})$; we assign the energy density stored in the magnetic field is $\frac{1}{2}\boldsymbol{B}\cdot\boldsymbol{H}$ (it is often written this way round).

4 Electromagnetic Waves

Note: because there are more symmetries between them, it becomes more convenient to use $\boldsymbol{E}$ and $\boldsymbol{H}$ to describe these waves, rather than $\boldsymbol{E}$ and $\boldsymbol{B}$

4.1 Free Space

Maxwell's Equations become:

$$\nabla \cdot \boldsymbol{E} = 0 \tag{11}$$
$$\nabla \cdot \boldsymbol{H} = 0 \tag{12}$$
$$\nabla \times \boldsymbol{E} = -\mu_0 \dot{\boldsymbol{H}} \tag{13}$$
$$\nabla \times \boldsymbol{H} = \epsilon_0 \dot{\boldsymbol{E}} \tag{14}$$

Taking the curl of, say, -(13) and using $\nabla \times (\nabla \times \boldsymbol{F}) = \nabla(\nabla \cdot \boldsymbol{F}) - \nabla^2 \boldsymbol{F}$, we obtain:

$$\begin{aligned} \nabla^2 \boldsymbol{E} &= \mu_0 \nabla \times \dot{\boldsymbol{H}} \\ &= \mu_0 \epsilon_0 \ddot{\boldsymbol{E}} \end{aligned}$$

which is the wave equation for $c = 1/\sqrt{\mu_0 \epsilon_0}$. Let there be light.
Similarly, taking the curl of -(14) gives:

$$\begin{aligned} \nabla^2 \boldsymbol{H} &= -\epsilon_0 \nabla \times \dot{\boldsymbol{E}} \\ &= \mu_0 \epsilon_0 \ddot{\boldsymbol{H}} \end{aligned}$$

giving an identical wave equation.

4.1.1 Plane waves

Plane waves have no variation perpendicular to their direction of travel. We define the vector $\boldsymbol{k}$ to be in the direction of travel, with magnitude $2\pi/\lambda$ where λ is the wavelength. Also, the equation $\nabla e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = i\boldsymbol{k}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$ will be important.
We can write a plane wave $\boldsymbol{E}$ field as $\boldsymbol{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$ (where $\boldsymbol{E}_0 \cdot \boldsymbol{k} \equiv 0$), and so

from (13) and $\boldsymbol{\nabla} \times (\phi \boldsymbol{F}) = \boldsymbol{\nabla}\phi \times \boldsymbol{F} + \phi \boldsymbol{\nabla} \times \boldsymbol{F}$,

$$\begin{aligned}
-\mu_0 \dot{\boldsymbol{H}} &= i\boldsymbol{k} \times \boldsymbol{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \\
\Rightarrow \boldsymbol{H} &= \frac{k}{\mu_0 \omega} \hat{\boldsymbol{k}} \times \boldsymbol{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \\
\Rightarrow \boldsymbol{H}_0 &= \frac{1}{\mu_0 c} \hat{\boldsymbol{k}} \times \boldsymbol{E}_0 \\
&= \sqrt{\frac{\epsilon_0}{\mu_0}} \hat{\boldsymbol{k}} \times \boldsymbol{E}_0 \\
&= \frac{1}{Z_0} \hat{\boldsymbol{k}} \times \boldsymbol{E}_0
\end{aligned}$$

where $Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$, *the impedance of free space*, is the ratio of the magnitudes of the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields in a vacuum. In a general material this ratio is the general impedance $Z = \sqrt{\frac{\mu\mu_0}{\epsilon\epsilon_0}}$.

4.2 Energy Transfer

4.2.1 Work on charges

If we briefly suppose that there are actually charges present, then there is not only potential energy stored in the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields of the wave itself (and energy *transfer* as the wave propagates), but also kinetic energy as the $\boldsymbol{E}$ field does work on the charges present. The power of this interaction is:

$$\begin{aligned}
P_{charges} &= \int_{\mathbb{R}^3} dq \boldsymbol{E} \cdot \boldsymbol{v} \\
&= \int_{\mathbb{R}^3} \boldsymbol{E} \cdot \rho \boldsymbol{v} d\tau \\
&= \int_{\mathbb{R}^3} \boldsymbol{E} \cdot \boldsymbol{J} d\tau
\end{aligned}$$

4.2.2 The Poynting Vector

Consider the vector $\boldsymbol{N} = \boldsymbol{E} \times \boldsymbol{H}$ and its divergence:

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{N} &= \boldsymbol{\nabla} \cdot (\boldsymbol{E} \times \boldsymbol{H}) \\
&= \boldsymbol{H} \cdot \boldsymbol{\nabla} \times \boldsymbol{E} - \boldsymbol{E} \cdot \boldsymbol{\nabla} \times \boldsymbol{H} \\
&= \boldsymbol{H} \cdot \left(-\mu_0 \dot{\boldsymbol{H}}\right) - \boldsymbol{E} \cdot (\boldsymbol{J} + \epsilon_0 \boldsymbol{E}) \\
&= -\frac{\partial}{\partial t}\left(\frac{1}{2}\mu_0 \boldsymbol{H} \cdot \boldsymbol{H}\right) - \frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon_0 \boldsymbol{E} \cdot \boldsymbol{E}\right) - \boldsymbol{E} \cdot \boldsymbol{J}
\end{aligned}$$

We can therefore interpret the Poynting vector $\boldsymbol{N}$ as a power flux per unit volume; in free space we simply have $\boldsymbol{J} = 0$. We see that it has magnitude $|\boldsymbol{N}| = |\boldsymbol{E}||\boldsymbol{H}| = |\boldsymbol{E}|^2/Z$. Like the fields themselves, the Poynting vector varies in magnitude along a wave, but unlike the fields the Poynting vector is always in the same direction:

$$\begin{aligned}
\boldsymbol{N} &= \boldsymbol{E} \times \boldsymbol{H} \\
&= \Re \boldsymbol{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \times \Re\left(\frac{1}{Z_0}\hat{\boldsymbol{k}} \times \boldsymbol{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\right) \\
&= \frac{|\boldsymbol{E}_0|^2}{Z_0}\cos^2\left(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t\right)\hat{\boldsymbol{k}}
\end{aligned}$$

where we see that the rms value of $|\boldsymbol{N}|$ is $|\boldsymbol{E}|^2/2Z$.

The Poynting vector can be used to quantify the radiation pressure $\boldsymbol{R}$ exerted by electromagnetic radiation, using the relationship for light $E = \boldsymbol{p}c$. This pressure is given by:

$$\begin{aligned}
\boldsymbol{R} &= \frac{1}{A}\frac{d\boldsymbol{p}}{dt} \\
&= \frac{1}{Ac}\frac{dE}{dt} \\
&= \frac{\boldsymbol{N}}{c}
\end{aligned}$$

4.3 Plasmas and Metals

Now we consider what happens when EMR is present within materials, ignoring boundaries and also net charge which is probably 0. Maxwell's Equations become:

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{E} &= 0 \\
\boldsymbol{\nabla} \cdot \boldsymbol{H} &= 0 \\
\boldsymbol{\nabla} \times \boldsymbol{E} &= -\mu\mu_0 \dot{\boldsymbol{H}} \\
\boldsymbol{\nabla} \times \boldsymbol{H} &= \boldsymbol{J}_f + \epsilon\epsilon_0 \dot{\boldsymbol{E}}
\end{aligned}$$

where we are supposing that there is no net charge of any kind anywhere (or alternatively that ϵ is constant).

4.3.1 Plasmas

Here we suppose that we have a sparse concentration of electrons which are all vaguely in the vicinity of a parent ion, and that this concentration is so small that $\boldsymbol{J}_f = 0$. Oh and $\mu = 1$ because everything is so hot that it'd be past everything's Curie temperature anyway. We then have:

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\mu_0 \dot{\boldsymbol{H}} \tag{15}$$

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \epsilon\epsilon_0 \dot{\boldsymbol{E}} \tag{16}$$

and

$$\boldsymbol{\nabla}^2 \boldsymbol{E} = \mu_0 \epsilon\epsilon_0 \ddot{\boldsymbol{E}}$$

and likewise for $\boldsymbol{H}$. To find what the solutions to this look like, we therefore need to find out what ϵ is.

First let's look at the Lorentz force on the particle:

$$m_e \ddot{\boldsymbol{r}} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

Think about the magnitude of the two terms of the force. $\boldsymbol{H}$ is smaller than $\boldsymbol{E}$ by a factor of Z, so $\boldsymbol{B}$ is smaller by a factor of c. Thus when $v << c$, as we will assume, the magnetic force makes very little contribution and we have:

$$m_e \ddot{\boldsymbol{r}} = q\boldsymbol{E}$$

If we are considering the response of electrons in an oscillating electric field $\boldsymbol{E} = \boldsymbol{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}$, we obtain:

$$\boldsymbol{r} = \frac{e}{m_e \omega^2} \boldsymbol{E}$$

If there are N electrons per unit volume, the $\boldsymbol{P}$ that this instantaneous displacement induces is $-Ne\boldsymbol{r}$, which we can set equal to $\chi_e \epsilon_0 \boldsymbol{E}$ to give:

$$\begin{aligned} \chi_e &= -\frac{Ne^2}{\epsilon_0 m_e \omega^2} \\ &= -\frac{\omega_p^2}{\omega^2} \end{aligned}$$

where $\omega_p^2 = \frac{Ne^2}{\epsilon_0 m_e} \approx 18^2 \pi^2 N$, which gives an easy expression for the frequency itself: $9\sqrt{N/m^{-3}}$. We can now find the relative permittivity, refractive index and dispersion relation of the plasma:

$$\begin{aligned} \Rightarrow \epsilon &= 1 + \chi_e = 1 - \frac{\omega_p^2}{\omega^2} \\ \Rightarrow n &= \sqrt{\epsilon} = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \\ \Rightarrow \omega &= \frac{kc}{n} = kc \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{-1/2} \end{aligned}$$

If $\omega < \omega_p$, we find that $\chi_e < -1$, $\epsilon < 0$ and $n \in \mathbb{I}$; this corresponds to the field oscillating at such a low magnitude that the dipoles induced get sooo large that they *more than cancel out* the original $\boldsymbol{E}$ field. You may remember that we had earlier the "wave" equation $\boldsymbol{\nabla}^2\boldsymbol{E} = \mu\mu_0\epsilon\epsilon_0\ddot{\boldsymbol{E}}$, but if ϵ is negative then the solutions are exponential (decays, to be physical). Setting $\epsilon = 1 - \omega_p^2/\omega^2 = -\beta^2$ (and $\mu = 1$), we have $\boldsymbol{\nabla}^2\boldsymbol{E} = -\frac{\beta^2}{c^2}\ddot{\boldsymbol{E}}$, and solutions like $\boldsymbol{E} = \boldsymbol{E}_0 e^{-\frac{\omega\beta}{c}\hat{\boldsymbol{k}}\cdot\boldsymbol{r} - i\omega t}$. This is no ordinary wave; this is an *evanescent* wave, which exponentially decays with distance into the plasma (supposing we have some kind of transmitter within the plasma, or a boundary between a vacuum and a plasma but then we would have to modify $\boldsymbol{\nabla}\cdot\boldsymbol{E} = 0$) with decay constant:

$$\delta = \frac{c}{\omega\beta} = \frac{c}{\omega\sqrt{\frac{\omega_p^2}{\omega^2} - 1}}$$

which clearly depends a lot on just how much lower ω is than ω_p.

What about the $\boldsymbol{H}$ field? Well going back to (16) and positing a similar functional form we obtain:

$$\begin{aligned} -\frac{\omega\beta}{c}\hat{\boldsymbol{k}} \times \boldsymbol{E}_0 &= i\omega\mu_0\boldsymbol{H}_0 \\ \Rightarrow \boldsymbol{H}_0 &= \frac{i\beta}{\mu_0 c}\hat{\boldsymbol{k}} \times \boldsymbol{E}_0 \\ &= \frac{i\beta}{Z_0}\hat{\boldsymbol{k}} \times \boldsymbol{E}_0 \end{aligned}$$

Now the fields are $\pi/2$ out of phase! As a consequence, the $\boldsymbol{N}$ vector becomes:

$$\begin{aligned} \boldsymbol{N} &= \Re\boldsymbol{E} \times \Re\boldsymbol{H} \\ &= -\boldsymbol{E}_0 \times \left(\hat{\boldsymbol{k}} \times \boldsymbol{E}_0\right)\frac{\beta}{Z_0}\cos(\omega t)\sin(\omega t)e^{-2\frac{\omega\beta}{c}\hat{\boldsymbol{k}}\cdot\boldsymbol{r}} \\ &= |\boldsymbol{E}_0|^2\hat{\boldsymbol{k}}\frac{\beta}{2Z_0}\sin(2\omega t)e^{-2\frac{\omega\beta}{c}\hat{\boldsymbol{k}}\cdot\boldsymbol{r}} \end{aligned}$$

which has a time average of 0 — on average no energy is being transferred, though it is sloshing back and forth a lot as it is withdrawn and deposited into the fields.

4.3.2 Metals

In metals, it is usually a good approximation that $\boldsymbol{J}_f = \sigma\boldsymbol{E}$, and we obtain:

$$\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{E} &= -\mu\mu_0\dot{\boldsymbol{H}} \\ \boldsymbol{\nabla} \times \boldsymbol{H} &= \sigma\boldsymbol{E} + \epsilon\epsilon_0\dot{\boldsymbol{E}} \end{aligned}$$

If we then try the same thing as at the beginning of this section, we obtain:

$$\boldsymbol{\nabla}^2 \boldsymbol{E} = \mu\mu_0 \left(\sigma \dot{\boldsymbol{E}} + \epsilon\epsilon_0 \ddot{\boldsymbol{E}} \right) \tag{17}$$

and an identical equation for $\boldsymbol{H}$. Here we do not neglect the σ — in fact we will see that it dominates. Using (15) and positing complex exponential solutions, we obtain:

$$\begin{aligned} \boldsymbol{\nabla}^2 \boldsymbol{E} &= \mu\mu_0 \left(-i\omega\sigma \boldsymbol{E} - \omega^2 \epsilon\epsilon_0 \boldsymbol{E} \right) \\ &= -\omega^2 \mu\mu_0 \epsilon_0 \left(\epsilon + \frac{i\sigma}{\omega\epsilon_0} \right) \boldsymbol{E} \end{aligned}$$

If we were dealing with insulators, whose dielectric constants behave as we would expect, the term proportional to σ would be 0. In a metal σ is (fortunately) not 0, so what we do is define an "effective dielectric constant" ϵ', equal to $\epsilon + \frac{i\sigma}{\omega\epsilon_0}$, so that we can analyse EMR in a metal like we would in a dielectric. However we haven't really analysed it in a dielectric so what was the point.

Anyway. In a metal it is usually the case that the conductivity is so high that the real part of ϵ' is negligible and it becomes essentially imaginary. We are therefore left with:

$$\boldsymbol{\nabla}^2 \boldsymbol{E} = -i\omega\mu\mu_0\sigma \boldsymbol{E}$$

So how does the $\boldsymbol{E}$ field vary spatially? or in other words, what is its $\boldsymbol{k}$? In order for the above equation to be satisfied, we see that the field must be given by:

$$\begin{aligned} \boldsymbol{E} &= \boldsymbol{E}_0 \exp\left(-\frac{1-i}{\sqrt{2}} \sqrt{\omega\mu\mu_0\sigma} \hat{\boldsymbol{k}} \cdot \boldsymbol{r} \right) e^{-i\omega t} \\ &= \boldsymbol{E}_0 \exp\left(-\hat{\boldsymbol{k}} \cdot \frac{\boldsymbol{r}}{\delta} \right) \exp\left(i\hat{\boldsymbol{k}} \cdot \frac{\boldsymbol{r}}{\delta} \right) e^{-i\omega t} \end{aligned}$$

where $\delta = \sqrt{\frac{2}{\sigma\omega\mu\mu_0}}$. This is technically still a wave, since there is some complex spatial oscillation, but it decays equally as rapidly as it oscillates. We see therefore, that if we have some EMR incident on a conducting surface [and we neglect all sorts of things, probably reasonably] then it will only get a distance δ into the wire before it is attenuated by a factor of e. As such δ is often therefore called the *skin depth* of the material, and for copper at 100MHz δ is about 6.5μm, motivating the thin connotation of the name!

As for the $\boldsymbol{H}$ field, we have again:

$$\begin{aligned} i\omega\mu_0\boldsymbol{H}_0 &= -\frac{1-i}{\delta}\hat{\boldsymbol{k}}\times\boldsymbol{E}_0 \\ \Rightarrow \boldsymbol{H}_0 &= \frac{1+i}{\sqrt{2}}\frac{\sqrt{2}}{\delta\omega\mu_0}\hat{\boldsymbol{k}}\times\boldsymbol{E}_0 \end{aligned}$$

where the important takeaway is that the $\boldsymbol{H}$ field is $\pi/4$ out of phase with the $\boldsymbol{E}$ field. The $\boldsymbol{N}$ vector now has an interesting form:

$$\begin{aligned} \boldsymbol{N} &= \boldsymbol{E}_0 \times \left(\hat{\boldsymbol{k}}\times\boldsymbol{E}_0\right)\frac{\sqrt{2}}{\delta\omega\mu_0}\cos(\omega t)\cos\left(\omega t+\frac{\pi}{4}\right)\exp\left(-2\hat{\boldsymbol{k}}\cdot\frac{\boldsymbol{r}}{\delta}\right) \\ &= |\boldsymbol{E}_0|^2\hat{\boldsymbol{k}}\frac{\sqrt{2}}{\delta\omega\mu_0}\left[\frac{1}{2}\cos\left(2\omega t+\frac{\pi}{4}\right)+\frac{1}{2\sqrt{2}}\right]\exp\left(-2\hat{\boldsymbol{k}}\cdot\frac{\boldsymbol{r}}{\delta}\right) \end{aligned}$$

which does have a net $\boldsymbol{k}$-ward direction, but this also oscillates and decays, occasionally directed opposite to $\boldsymbol{k}$.

These results have significant consequences for the movement of AC currents in wires. If a wire carries an AC current, there will be a time-varying $\boldsymbol{E}$ field directed up and down the wire to drive the current. It turns out that the electric field within a wire can be described (roughly) as like the electric field of an EM wave when it is incident on the wire. If the current is propagating along the x-axis the $\boldsymbol{E}$ field along the z-axis (perpendicular and into the wire, so that $z=0$ on the surface and z increases towards the centre) can be thought of as:

$$\boldsymbol{E}_0\exp\left(-\frac{z}{\delta}\right)\exp\left(i\frac{z}{\delta}\right)e^{-i\omega t}$$

with a corresponding $\boldsymbol{J}$ of:

$$\boldsymbol{J}_0\exp\left(-\frac{z}{\delta}\right)\exp\left(i\frac{z}{\delta}\right)e^{-i\omega t}$$

The total current in the wire of radius a is then given by:

$$\begin{aligned} I &= J_0e^{-i\omega t}\int_0^a e^{\frac{-1+i}{\delta}z}2\pi(a-z)dz \approx 2\pi J_0e^{-i\omega t}\int_0^a e^{\frac{-1+i}{\delta}z}adz \\ &= 2\pi aJ_0e^{-i\omega t}\frac{\delta}{-1+i}\left(e^{\frac{-1+i}{\delta}a}-1\right)\approx 2\pi aJ_0e^{-i\omega t}\frac{\delta}{1-i} \\ &= \pi aJ_0\delta(1+i)e^{-i\omega t} \\ &= \sqrt{2}\pi aJ_0\delta e^{-i(\omega t-\frac{\pi}{4})} \\ \Rightarrow \langle I^2\rangle &= (\pi aJ_0\delta)^2 \end{aligned}$$

The power/unit length can also be calculated, along with the effective resistance per unit length. The takeaway is that said resistance is the same as if all the current were travelling in a thin shell of thickness δ.

5 Waveguides

Until now (in life), it is assumed that for a wire the whole thing is at the same voltage. However, if the Fourier components of an incident wave are at a very high frequency, the "wavelength" of the signal will be less than the length of the wire, and the voltage will vary along the wire (as will the current). Alternatively, the wire may just be really long (e.g. kilometres), and so the same effect occurs. Wires (and other systems) capable of transporting these voltages and currents are called *transmission lines*; The term *waveguide* refers to carriers where the wavelength is much smaller compared to the dimension.

5.1 Transmission Lines

An illustrative setup consists of two (superconducting) parallel wires, stretching a long way into the distance. There will be some parasitic inductance and capacitance due to this setup; we say that there is L and C **per unit length**, so the equations will look a bit dimensionally weird. Consider a small segment of wire, of length dz. This will have an inductance $L\ dz$, so the voltage at $z + dz$ will be given by:

$$V(z + dz) = V(z) - L\ dz\frac{\partial I}{\partial t}$$

Also, the wire segment will have a capacitance $C\ dz$, so the current at $z + dz$ will be:

$$I(z + dz) = I(z) - C\ dz\frac{\partial V}{\partial t}$$

From these, we can obtain the pair of equations:

$$\frac{\partial V}{\partial z} = -L\frac{\partial I}{\partial t} \qquad\qquad \frac{\partial I}{\partial z} = -C\frac{\partial V}{\partial t}$$

from which we can obtain:

$$\frac{\partial^2 V}{\partial z^2} = LC\frac{\partial^2 V}{\partial z^2} \qquad\qquad \frac{\partial^2 I}{\partial z^2} = LC\frac{\partial^2 I}{\partial z^2}$$

describing a wave travelling at $v = \frac{1}{\sqrt{LC}}$. For a pair of parallel wires, it can be calculated that:

$$L = \frac{\mu_0}{\pi} \ln\left(\frac{D}{a}\right) \qquad\qquad C = \frac{\pi\epsilon_0}{\ln\left(\frac{D}{a}\right)}$$

so in fact $v = \frac{1}{\sqrt{\mu_0\epsilon_0}} = c$; the voltage and current waveforms move along the wire at the speed of light (assuming everything's ideal).

Transmission lines are really the domain of engineering, so we write for a Fourier component:

$$V(z,t) = V_0 e^{-j(kz-\omega t)}$$

and similarly for $I(z,t)$, where $\omega = ck$. We then have, from one of the original differential equations:

$$-jkV = -jL\omega I$$
$$\Rightarrow Z \equiv \frac{V}{I} = Lc = \sqrt{\frac{L}{C}}$$

Z is known as the *characteristic impedance* of the transmission line.

Most transmission lines have a beginning and end, rather than being infinitely long. In other words, z can only vary between 0 and, say, l. We can think about placing components at the end of transmission lines as giving a boundary condition to the wave equation at $z = l$. For example, we might have a resistor of resistance/impedance Z at the end - this corresponds to imposing $V = IZ$ at the terminal, but this is already true, so no extra restrictions are placed on it. Such "impedance-matched" resistors lead the line to behave *as if* it were infinite. Another example would be a free or short-circuited end, i.e. where it is imposed that $I(l,t) = 0$ or $V(l,t) = 0$ respectively. However, because $V = IZ$ at all points on the wire, we have in both cases that both $V(l,t)$ and $I(l,t)$ are equal to 0 at the end. This sets up standing wave solutions, but more practically means that if a pulse is sent from the start of the line, it will have to be somehow *reflected* from the other end in order to maintain the boundary condition.

5.1.1 Reflections on Transmission Lines

A signal with a range of Fourier components $V_i = V_{i0}e^{-j(kz-\omega t)}$ is sent out along a transmission line, creating a current $I_i = \frac{V_{i0}}{Z}e^{-j(kz-\omega t)}$. It reaches the end of the line (previously $z = l$, but here we will define that to be $z = 0$ to give an easier sense of what's coming from where), where it encounters an impedance Z_t, and a reflected signal is produced, with components $V_r = V_{r0}e^{-j(-kz-\omega t)}$ and currents $I_r = -\frac{V_{r0}}{Z}e^{-j(-kz-\omega t)}$ (as they are headed in the opposite direction). We have:

$$\begin{aligned} Z_t &= \frac{V_i(0,t) + V_r(0,t)}{I_i(0,t) + I_r(0,t)} \\ &= \frac{V_{i0}e^{-j(-\omega t)} + V_{r0}e^{-j(-\omega t)}}{\frac{V_{i0}}{Z}e^{-j(-\omega t)} - \frac{V_{r0}}{Z}e^{-j(-\omega t)}} \\ &= \frac{Z + rZ}{1 - r} \end{aligned}$$

where $r \equiv V_{r0}/V_{i0}$. Incidentally, from this we obtain that:

$$r = \frac{Z_t - Z}{Z_t + Z}$$

confirming the above analysis which suggested that if $Z_t = Z$ there would be no reflection. Z_t can be expressed in terms of the input impedence Z_i, defined as $V(-l,t)/I(-l,t)$. We thus have:

$$\begin{aligned} Z_i &= \frac{V_i(-l,t) + V_r(-l,t)}{I_i(-l,t) + I_r(-l,t)} \\ &= \frac{V_{i0}e^{-j(-kl-\omega t)} + V_{r0}e^{-j(kl-\omega t)}}{\frac{V_{i0}}{Z}e^{-j(-kl-\omega t)} - \frac{V_{r0}}{Z}e^{-j(kl-\omega t)}} \\ &= Z\frac{e^{jkl} + re^{-jkl}}{e^{jkl} - re^{-jkl}} \\ &= Z\frac{(Z_t + Z)e^{jkl} + (Z_t - Z)e^{-jkl}}{(Z_t + Z)e^{jkl} - (Z_t - Z)e^{-jkl}} \\ &= Z\frac{Z_t\cos(kl) + jZ\sin(kl)}{Z\cos(kl) + jZ_t\sin(kl)} \end{aligned}$$

A particularly interesting case is for $kl = \pi/2$, i.e. a *quarter-wave line*. In this case (i.e. for this l and this frequency component), we have $Z_i = \frac{Z^2}{Z_t}$. This probably means that there is no reflection at the input terminal, and all of the "source input" is absorbed by the line.

5.2 Waveguides

The simplest waveguides consist of metal rectangular tubes, within which electromagnetic waves propagate, though these waves have subtle differences to those previously seen. Let the waveguide have a width a and a height b, with the wave propagating in the z-direction. We then have simply the equations (11-14) which describe any electromagnetic wave, but now we have some specific boundary conditions.

The walls of the waveguide are assumed not just to be conductors, but *perfect* conductors, i.e. with an infinite conductivity. If there were any $\boldsymbol{E}$ field along the surface then this would therefore lead to a current which would instantaneously cancel out the $\boldsymbol{E}$ field. If you prefer, from the metal's perspective, the charges within it move so quickly that from their perspective the situation is essentially static, and in electrostatic situations the curl is zero so there is no parallel component at the walls. Secondly, there is apparently no magnetic field inside the conductor, so the normal component of $\boldsymbol{H}$ at the walls must be 0.

There are two types of modes that a waveguide can support. One, the TE_{mn} modes, have the $\boldsymbol{E}$ field everywhere transverse to the z-direction – that is, $\boldsymbol{E}_z = 0$. These modes must have $\boldsymbol{E}_x(x, 0, z, t) = \boldsymbol{E}_x(x, b, z, t) = 0$, and so $\boldsymbol{E}_x$ is made proportional to $\sin(n\pi y/b)$. Similarly, $\boldsymbol{E}_y$ is made proportional to $\sin(m\pi x/a)$. So that they represent a travelling wave, both are made proportional to $\cos(k_z z - \omega t)$. We therefore have:

$$\boldsymbol{E}_x = A_x(x) \sin\left(\frac{n\pi y}{b}\right) \cos(k_z z - \omega t)$$
$$\boldsymbol{E}_y = A_y(y) \sin\left(\frac{m\pi x}{a}\right) \cos(k_z z - \omega t)$$

Imposing $\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0$ gives:

$$\frac{\partial A_x}{\partial x} \sin\left(\frac{n\pi y}{b}\right) + \frac{\partial A_y}{\partial y} \sin\left(\frac{m\pi x}{a}\right) = 0$$

We therefore have, for example:

$$A_x(x) = A_0 \frac{an}{\pi} \cos\left(\frac{m\pi x}{a}\right)$$
$$A_y(y) = -A_0 \frac{bm}{\pi} \cos\left(\frac{n\pi y}{b}\right)$$

which can be easily seen to work, giving:

$$\boldsymbol{E}_x = A_0 \frac{an}{\pi} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \cos(k_z z - \omega t)$$
$$\boldsymbol{E}_y = -A_0 \frac{bm}{\pi} \cos\left(\frac{n\pi y}{b}\right) \sin\left(\frac{m\pi x}{a}\right) \cos(k_z z - \omega t)$$

From this field, one can calculate the $\boldsymbol{H}$ field, for the $\mathrm{TE_{mn}}$ mode:

$$\boldsymbol{H}_x = \frac{A_0 k_z b m}{\mu_0 \omega \pi} \cos\left(\frac{n\pi y}{b}\right) \sin\left(\frac{m\pi x}{a}\right) \cos(k_z z - \omega t)$$
$$\boldsymbol{H}_y = \frac{A_0 k_z a n}{\mu_0 \omega \pi} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \cos(k_z z - \omega t)$$
$$\boldsymbol{H}_z = -\frac{A_0}{\mu_0 \omega}\left(\frac{m^2 b}{n} + \frac{n^2 a}{b}\right) \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \sin(k_z z - \omega t)$$

which gives $\nabla \cdot \boldsymbol{H} = 0$ on sight, and we also have $\boldsymbol{H}_x(x = 0) = \boldsymbol{H}_x(x = a) = \boldsymbol{H}_y(y = 0) = \boldsymbol{H}_y(y = b) = 0$ as is apparently required. Importantly, here we see that $\boldsymbol{H}$ has a component in the direction of travel, unlike for a regular EM wave; we say that the waveguide does not support an EM wave.

Both these fields also satisfy the wave equation, for:

$$\frac{\omega^2}{c^2} = \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} + k_z^2$$

Note that k_z is only real (i.e. the wave will only propagate along the waveguide) if:

$$\omega \geq c\sqrt{\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}}$$

The ω at equality is known as the *cutoff*, and the *cutoff frequency* f_c is given by:

$$f_c = \frac{c}{2\pi}\sqrt{\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}} = \frac{c}{2}\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$

A similar analysis can be done for the $\mathrm{TM_{mn}}$ modes, but we now set $\boldsymbol{H}_z = 0$ and impose e.g. $\boldsymbol{H}_x(x = 0) = \boldsymbol{H}_x(x = a) = 0$ as $\boldsymbol{H}$ can have no perpendicular component on the metal surface. We obtain:

$$\boldsymbol{H}_x = A_0 \frac{an}{\pi} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \cos(k_z z - \omega t)$$
$$\boldsymbol{H}_y = -A_0 \frac{bm}{\pi} \sin\left(\frac{n\pi y}{b}\right) \cos\left(\frac{m\pi x}{a}\right) \cos(k_z z - \omega t)$$
$$\boldsymbol{H}_z = 0$$
$$\boldsymbol{E}_x = -\frac{A_0 k_z b m}{\epsilon_0 \omega \pi} \sin\left(\frac{n\pi y}{b}\right) \cos\left(\frac{m\pi x}{a}\right) \cos(k_z z - \omega t)$$
$$\boldsymbol{E}_y = -\frac{A_0 k_z a n}{\epsilon_0 \omega \pi} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \cos(k_z z - \omega t)$$
$$\boldsymbol{E}_z = -\frac{A_0}{\epsilon_0 \omega}\left(\frac{m^2 b}{a} + \frac{n^2 a}{b}\right) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin(k_z z - \omega t)$$

Note that although $\mathrm{TE_{mn}}$ modes exist for m or n being 0, TM_{mn} modes do not, as for either $m = 0$ or $n = 0$ all components of both $\boldsymbol{E}$ and $\boldsymbol{H}$ go to 0.