Relativity

Xander Byrne

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1 Special Relativity

The Lorentz Transformations:

$$\mathrm{d}x' = \gamma(\mathrm{d}x - \beta\,\mathrm{d}ct) \qquad\qquad \mathrm{d}ct' = \gamma(\mathrm{d}ct - \beta\,\mathrm{d}x)$$

This transformation conserves the spacetime interval $\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$, ensuring that the speed of light is the same for all observers.

The *rapidity* $\psi \equiv \operatorname{artanh}\beta$, so:

$$\beta = \tanh\psi \qquad \gamma = \cosh\psi \qquad \gamma\beta = \sinh\psi$$

so the Lorentz transformations can be written:

$$\mathrm{d}x' = \cosh\psi\,\mathrm{d}x - \sinh\psi\,\mathrm{d}ct \qquad\qquad \mathrm{d}ct' = -\sinh\psi\,\mathrm{d}x + \cosh\psi\,\mathrm{d}ct$$

1.1 Length Contraction & Time Dilation

1.1.1 Length Contraction

See Figure 1. We know $\Delta x'_{OA} = \Delta x'_{OB} = L_0$, $\Delta ct_{OA} = 0$, we seek Δx_{OA}, and we do not know $\Delta ct'_{OA}$. We therefore use the equation which has the first three

$$\Delta x'_{OA} = \gamma(\Delta x_{OA} - \beta\Delta ct_{OA})$$

Substituting $\Delta ct_{OA} = 0$ and $\Delta x'_{OA} = L_0$, we find that the length of the rod in $\mathcal{S}$ is $\Delta x_{OA} = L_0/\gamma$: the rod appears longer in its rest frame ($\mathcal{S}'$) by a factor of γ than in a different frame $\mathcal{S}$.

1.1.2 Time Dilation

Similarly to $\Delta x'_{OA} = \Delta x'_{OB}$ in the length contraction example, we have in this case $\Delta ct_{CE} = \Delta ct_{CD}$. With $\Delta x'_{CD} = 0$, $\Delta ct'_{CD} = cT_0$, and Δx_{CD} unknown, we use one of the inverse Lorentz formulae:

$$\Delta ct_{CE} = \Delta ct_{CD} = \gamma(\Delta ct'_{CD} - \beta\underbrace{\Delta x'_{CD}}_{0}) = \gamma\Delta ct'_{CD} \Rightarrow \Delta t = \gamma T_0$$

to find that the lifetime in $\mathcal{S}$ is in fact *dilated* by a factor of γ.

The time between two events is always shortest in the frame in which they occur in the same place (if such a frame exists). This is a result of the spacetime interval $\Delta s^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$ being the same in all frames.

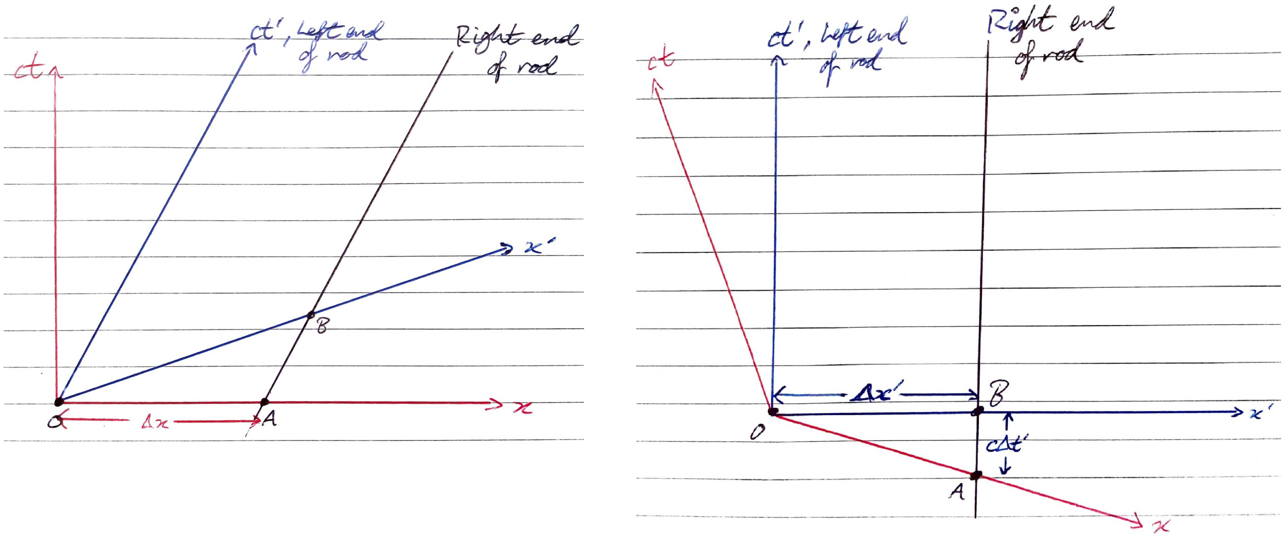

Figure 1 | Spacetime Diagrams of Length Contraction. (a) shows the setup as viewed by $\mathcal{S}$; (b) shows the setup as viewed by $\mathcal{S}'$

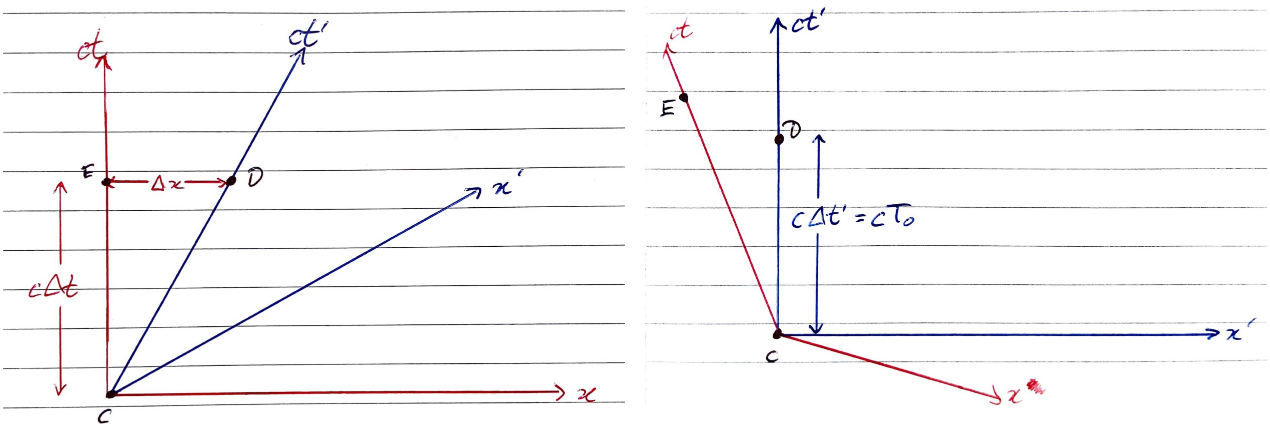

Figure 2 | Spacetime Diagrams of Time Dilation. (a) shows the setup as viewed by $\mathcal{S}$; (b) shows the setup as viewed by $\mathcal{S}'$

1.2 Doppler Effect

A source moving at speed v emits light of period T_0, as recorded in the source's rest frame $\mathcal{S}'$, and it passes an observer in $\mathcal{S}$ just as it emits a peak, which is immediately received (at $x = x' = ct = ct' = 0$). In $\mathcal{S}'$, a time T_0 later another peak is emitted back towards the observer; in $\mathcal{S}$, this time period is dilated, so from the point of view of an omniscient in the same frame as the observer, the second peak is emitted after a time γT_0. During this time, however, the source has also moved $v\gamma T_0$ away from the observer, and the light has to travel back this distance for the observer to see the second peak; this takes a further time $v\gamma T_0/c = \beta\gamma T_0$. Thus the total time period that elapses between the peaks arriving to the observer's telescope is:

$$T = \gamma T_0 + \beta\gamma T_0 = (1+\beta)\gamma T_0 = \sqrt{\frac{1+\beta}{1-\beta}}T_0 \qquad \Rightarrow \qquad f = \sqrt{\frac{1-\beta}{1+\beta}}f_0$$

We see that the frequency of light is reduced if the source is heading away (*receding*) from us.

1.3 Transformation of Velocities and Acceleration

To derive the transformation formulae, remember that velocities and accelerations in a given frame involve differentiation with respect to time *in that frame*:

$$u_x = \frac{\mathrm{d}x}{\mathrm{d}t} \qquad u'_x = \frac{\mathrm{d}x'}{\mathrm{d}t'}$$
$$a_x = \frac{\mathrm{d}u_x}{\mathrm{d}t} \qquad a'_x = \frac{\mathrm{d}u'_x}{\mathrm{d}t'}$$

and similarly in the y- and z-directions. Some illustrative derivations:

$$\begin{aligned}
u'_x &\equiv \frac{\mathrm{d}x'}{\mathrm{d}t'} = \frac{\gamma(\mathrm{d}x - v\,\mathrm{d}t)}{\gamma(\mathrm{d}t - \frac{v}{c^2}\,\mathrm{d}x)} = \frac{\mathrm{d}x/\mathrm{d}t - v}{1 - \frac{v}{c^2}\,\mathrm{d}x/\mathrm{d}t} = \frac{u_x - v}{1 - u_x v/c^2} \\
u'_y &= \frac{u_y}{\gamma(1 - u_x v/c^2)} \\
a'_x &\equiv \frac{\mathrm{d}u'_x}{\mathrm{d}t'} = \frac{1 - u_x v/c^2 + (u_x - v)v/c^2}{(1 - u_x v/c^2)^2}\,\mathrm{d}u_x \frac{1}{\gamma(\mathrm{d}t - \frac{v}{c^2}\,\mathrm{d}x)} = \frac{1 - v^2/c^2}{(1 - u_x v/c^2)} \frac{1}{\gamma(1 - u_x v/c^2)^2} \frac{\mathrm{d}u_x}{\mathrm{d}t} \\
&= \frac{a_x}{\gamma^3(1 - u_x v/c^2)^3} \\
a'_y &= \frac{a_y}{\gamma^2(1 - u_x v/c^2)^2} + \frac{u_y v}{c^2} \frac{a_x}{\gamma^2(1 - u_x v/c^2)^3}
\end{aligned}$$

In the final expression, there are two terms as u'_y depends on both u_y and u_x, which can both depend on t'. In each case, the inverse formula may be obtained by switching primed and unprimed variables and switching v for $-v$. Although the value of the measured acceleration depends on the frame (we don't just have $a'_x = a_x$ unless $v = 0$), if $\mathbf{a} = \mathbf{0}$, then $\mathbf{a}' = \mathbf{0}$.

1.4 Worldlines in Spacetime

The path of a particle through spacetime (a *worldline*) can be expressed in a given frame as $(x(t), y(t), z(t))$, or in a parametrised fashion, expressing t, x, y, z each as a function of a parameter: $(t(\zeta), x(\zeta), y(\zeta), z(\zeta))$. A useful parameter is "proper time" τ, which is the time as measured by an observer travelling along with the particle. The particle's frame is called the *instantaneous rest frame* (IRF, symbolised $\mathcal{F}$).

Two events on the worldline of the particle, at τ and $\tau + \mathrm{d}\tau$, by definition occur at the same place in $\mathcal{F}$, and so $\mathrm{d}s^2 = c^2\,\mathrm{d}\tau^2$. In $\mathcal{S}$, moving relative to $\mathcal{F}$ at velocity $\boldsymbol{u}$, the two events are separated by a time $\mathrm{d}t$ and a spatial vector $(\mathrm{d}x\,, \mathrm{d}y\,, \mathrm{d}z) = \boldsymbol{u}\,\mathrm{d}t$. As $\mathrm{d}s^2$ is invariant,

$$\begin{aligned}
c^2\,\mathrm{d}\tau^2 &= c^2\,\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 = c^2\,\mathrm{d}t^2\left(1 - \frac{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}{c^2\,\mathrm{d}t^2}\right) \\
&= c^2\,\mathrm{d}t^2\left(1 - \frac{v^2}{c^2}\right) \qquad \Rightarrow \qquad \boxed{\mathrm{d}\tau = \mathrm{d}t\,/\gamma}
\end{aligned}$$

and so the time between two events is shortest in the frame where they occur at the same point. This enables any observer to calculate proper times between events:

$$\Delta\tau = \int \mathrm{d}\tau = \int \frac{\mathrm{d}t}{\gamma} = \int \sqrt{1 - \frac{v(t)^2}{c^2}}\,\mathrm{d}t$$

Suppose an accelerometer onboard a rocket reads $f(\tau)$. The acceleration of the rocket as viewed in another frame $\mathcal{S}$ can be deduced using the acceleration transformation formulae given above, with $x = u_x = \mathbf{0}$. Taking either $\mathcal{F}$ to play the role of the primed frame, using $u_x = v$ (this is the speed of the particle of interest, which in this case is the origin of $\mathcal{F}$), or using the inverse formula and use $u_x = 0$; either way we obtain $a_x = f(\tau)/\gamma^3$, or:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \left(1 - \frac{v^2}{c^2}\right)^{3/2} f(\tau) \qquad \Rightarrow \qquad \frac{\mathrm{d}v}{\mathrm{d}\tau} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}v}{\mathrm{d}t} = \left(1 - \frac{v(\tau)^2}{c^2}\right) f(\tau)$$

This ODE can be simplified by using the rapidity: substituting $v = c\tanh\psi$,

$$c\frac{\mathrm{d}\psi}{\mathrm{d}\tau} = f(\tau) \qquad \Rightarrow \psi(\tau) = \frac{1}{c}\int^{\tau} f(\xi)\,\mathrm{d}\xi$$

where the lower bound is fixed by the initial velocity. We can also find $t(\tau)$ and $x(\tau)$:

$$\begin{aligned}
\frac{\mathrm{d}t}{\mathrm{d}\tau} &= \gamma = \frac{1}{1 - \tanh^2\psi} = \cosh\psi & &\Rightarrow t(\tau) = \int^{\tau} \cosh\psi(\xi)\,\mathrm{d}\xi \\
\frac{\mathrm{d}x}{\mathrm{d}\tau} &= \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = v\gamma = c\tanh\psi\cosh\psi = c\sinh\psi & &\Rightarrow x(\tau) = c\int^{\tau} \sinh\psi(\xi)\,\mathrm{d}\xi
\end{aligned}$$

where again the lower bounds are fixed by initial conditions. We can then find $x(t)$.

2 Manifolds

2.1 Submanifolds

Subsets of points in a manifold $\mathcal{M}$ define submanifolds. A curve is a one-dimensional submanifold, parametrised by a single coordinate u. If $\mathcal{M}$ is charted by a coordinate system $(x^1, x^2, ..., x^N)$, the curve can be specified by setting each x^a to be some function of u.

Higher-dimensional submanifolds are *surfaces*, requiring the coordinates x^a to be parametrised by as many parameters as the dimensionality of the surface.

If the dimensionality of the submanifold is just 1 less than the dimensionality of the manifold N, the submanifold is a *hypersurface*, and the $N-1$ parameters can be eliminated to give a single constraint on all the coordinates: e.g. $x^2 + y^2 + z^2 = 1$.

2.2 Coordinate Transformations

If I have a system of coordinates that chart $\mathcal{M}$, so that any point can be specified by an N-tuple $(x^1, x^2 \dots x^N)$ and you write yours as $(x'^1, x'^2 \dots x'^N)$, we can convert between the unprimed and primed coordinate systems – that is, find the set of N functions $x'^a(x^b)$ of N variables.

Using the chain rule and summation convention:

$$\mathrm{d}x'^a = \frac{\partial x'^a}{\partial x^b}\,\mathrm{d}x^b \qquad \mathrm{d}x^a = \frac{\partial x^a}{\partial x'^b}\,\mathrm{d}x^b$$

where we require the matrix $\left[\partial x^a / \partial x'^b\right]$ to be invertible.

2.3 Metric

In Euclidean space, the squared distance between two infinitesimally separated points is:

$$\mathrm{d}s^2 = \left(\mathrm{d}x^1\right)^2 + \left(\mathrm{d}x^2\right)^2 + \ldots + \left(\mathrm{d}x^N\right)^2 = \delta_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b$$

On a general manifold, the squared infinitesimal distance takes the form $\boxed{\mathrm{d}s^2 = g_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b}$, where g_{ab} depends not only on the coordinate system used, but generally also on the location in the manifold, and so can be written as a function of the x^a. Wlog we can choose g_{ab} to be symmetric in its indices; the metric is therefore specified for a given coordinate system by $\frac{1}{2}N(N+1)$ independent functions of position (of the x^a).

The actual, true distance between these two neighbouring points does not depend on the coordinate system in which this distance is measured in. Therefore, in a different coordinate system, we find:

$$\mathrm{d}s^2 = g_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b = g_{ab}\frac{\partial x^a}{\partial x'^c}\,\mathrm{d}x'^c\,\frac{\partial x^b}{\partial x'^d}\,\mathrm{d}x'^d = g'_{cd}\,\mathrm{d}x'^c\,\mathrm{d}x'^d \qquad \text{where} \qquad g'_{cd} = \frac{\partial x^a}{\partial x'^c}\frac{\partial x^b}{\partial x'^d}g_{ab}$$

Suppose we start with a known metric g_{ab}, and attempt to find a new coordinate system $x'^a(x^b)$ in which *at a particular point* the metric $g'_{cd} = \delta_{cd}$. It turns out that not only is there always enough freedom in a general coordinate transformation to do this, there is some "residual freedom": not only are there infinitely many coordinate systems which would locally have $g'_{cd} = \delta_{cd}$ but there are $\frac{1}{2}N(N-1)$ residual *degrees of freedom* for an N-dimensional manifold: six for $N = 4$, corresponding to the degrees of freedom in a general Lorentz transform (a boost and a rotation).

For *pseudo-Riemannian* manifolds, the metric can be chosen to locally take some other diagonal form, such as $\eta_{ab} = \mathfrak{diag}(1,-1,-1,-1)$, the *Minkowski metric*, which gives rise to the Minkowski spacetime interval $c^2\,\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$.

We can in fact go further and find a coordinate system in which the manifold locally looks even more like a Euclidean manifold, by imposing $\partial g'_{ab}/\partial x'^c = 0$[1]. This is always possible, but only just – there are no more residual degrees of freedom in our choice of coordinate transform.

We might wish to constrain $\partial^2 g'_{ab}/\partial x'^c \partial x'^d = 0$. However, the second derivatives of the coordinate transformation $\left(\partial^2 x^a / \partial x'^b \partial x'^c\right)$ are overdetermined by this constraint, so no such transformation exists. For a general manifold there exists no coordinate system in which all the $\partial^2 g'_{ab}/\partial x'^c \partial x'^d = 0$, even locally. This is due to the manifolds' intrinsic curvature.

3 Vector & Tensor Algebra

3.1 Scalars

Scalar fields assign a number to each point in the manifold. This assignment is to the *points themselves*, not to the set of coordinates describing them in a certain system, so we must have $\phi(x^a) = \phi'(x'^b)$ if x^a and x'^b denote the same point.

[1] Such coordinates are called *local Cartesian coordinates*

3.2 Vectors

Consider an N-dimensional manifold $\mathcal{M}$ and a point $P \in \mathcal{M}$. The set of all possible local vectors at P is an N-dimensional vector space, denoted $T_P(\mathcal{M})$ and pronounced the tangent space of $\mathcal{M}$ at P. Consider the operator:

$$\mathbf{v} = v^a \frac{\partial}{\partial x^a}$$

The set of operators of this form clearly obey all the axioms of a vector space (closure, linearity, scalar multiplicability etc.), and is said to form $T_P(\mathcal{M})$. This space therefore has a basis $\{\partial/\partial x^a\}$, with v^a being the components.

In a different coordinate system x'^b, the basis vectors are:

$$\frac{\partial}{\partial x'^a} = \frac{\partial x^b}{\partial x'^a}\frac{\partial}{\partial x^b}$$

But for the actual physical vector operator to be independent of the coordinate system, its components must transform in the opposite way:

$$v'^a = \frac{\partial x'^a}{\partial x^b} v^b$$

so that:

$$\mathbf{v} = v^a \frac{\partial}{\partial x^a} = \frac{\partial x^a}{\partial x'^b} v'^b \frac{\partial x'^c}{\partial x^a}\frac{\partial}{\partial x'^c} = \frac{\partial x'^c}{\partial x'^b} v'^b \frac{\partial}{\partial x'^c} = \delta^c_b v'^b \frac{\partial}{\partial x'^c} = v'^b \frac{\partial}{\partial x'^b}$$

Thus the components do what components are supposed to: be the coefficients of the basis vectors to form the vector in question. Any N-tuple which transforms in this way forms the components of a vector. For example, the components of the infinitesimal displacement vector:

$$\mathrm{d}x'^a = \frac{\partial x'^a}{\partial x^b}\,\mathrm{d}x^b$$

or the tangent vector to a curve $x^a = x^a(u)$ for a parameter u, $\mathrm{d}/\mathrm{d}u$, whose components $\mathrm{d}x^a/\mathrm{d}u$ transform like:

$$\frac{\mathrm{d}x'^a}{\mathrm{d}u} = \frac{\partial x'^a}{\partial x^b}\frac{\mathrm{d}x^b}{\mathrm{d}u}$$

3.3 Dual Vectors

The gradients of scalar fields have components:

$$X_a = \frac{\partial \phi}{\partial x^a}$$

Their components transform in the opposite way to the components of vectors, and in the same way as the basis vectors $\partial/\partial x^a$:

$$X'_a = \frac{\partial x^b}{\partial x'^a}\frac{\partial \phi}{\partial x^b} = \frac{\partial x^b}{\partial x'^a} X_b$$

N-tuples which transform in this way form the components of a *dual vector*, and they also obey the axioms of a vector space: the dual vector space $T^*_P(\mathcal{M})$. Mathematicians sometimes introduce $T_P^*(\mathcal{M})$ as a space of linear maps[2] from $T_P(\mathcal{M})$ to $\mathbb{R}$. Such a mapping is performed by *contracting* a dual vector with a vector, giving a quantity like $X_a v^a$ which transforms as a scalar:

$$X'_a v'^a = \frac{\partial x^b}{\partial x'^a} X_b \frac{\partial x'^a}{\partial x^c} v^c = \delta^b_c X_b v^c = X_b v^b$$

3.4 Tensor Fields

We can extend this to define an object which constitutes a mapping from k copies of $T_P^*(\mathcal{M})$ and l copies of $T_P(\mathcal{M})$, to $\mathbb{R}$. These mappings can vary over the manifold, taking copies of the tangent and dual tangent spaces at any point on it; this defines a *tensor field* $\mathbf{T}$, which is said to be of *type* (k, l) and of *rank* $k + l$.

The tensor's components are written as $T^{a...b}{}_{c...d}$, and they transform in the following way:

$$T'^{a...b}{}_{c...d} = \frac{\partial x'^a}{\partial x^p} \cdots \frac{\partial x'^b}{\partial x^q} \frac{\partial x^r}{\partial x'^c} \cdots \frac{\partial x^s}{\partial x'^d} T^{p...q}{}_{r...s}$$

The order of *all* of a tensor's indices matter, not just those of upstairs and downstairs individually; for example $T^a{}_{bc}{}^d{}_e \neq T_b{}^{ad}{}_{ce}$ in general.

Tensor-valued objects are useful because they enable the writing of equations which are true in whatever coordinate system one chooses. For example, if $T^a{}_b = S^a{}_b$ in a certain coordinate system, it must be true in all of them, the tensors are the same, and we can write $\mathbf{S} = \mathbf{T}$. Similarly, if all the components $T_{ab} = 0$ in one coordinate system, this must be true in all of them and $\mathbf{T} = \mathbf{0}$. This is to be compared with setting $\mathbf{u} = \mathbf{v}$ or $\mathbf{v} = \mathbf{0}$, where they are both vectors; each component satisfies the equality individually.

3.4.1 Inner and Outer Products

The outer product of two tensors $\mathbf{S}$ and $\mathbf{T}$ is denoted $\mathbf{S} \otimes \mathbf{T}$. For two rank-1 tensors (vectors) $\mathbf{u}$ and $\mathbf{v}$ with components u^a and v^a in some coordinate system, their tensor product $\mathbf{T} = \mathbf{u} \otimes \mathbf{v}$ has components $T^{ab} = u^a v^b$ in this system. $\mathbf{T}$ is a tensor[3], as its components transform in the right way:

$$T'^{ab} = u'^a v'^b = \frac{\partial x'^a}{\partial x^c} u^c \frac{\partial x'^b}{\partial x^d} v^d = \frac{\partial x'^a}{\partial x^c} \frac{\partial x'^b}{\partial x^d} T^{cd}$$

Any object which is the tensor product of two tensors is itself a tensor. Generally, $\mathbf{S} \otimes \mathbf{T} \neq \mathbf{T} \otimes \mathbf{S}$.

Contraction is the action of setting an upstairs index equal to a downstairs index and summing. For example, the tensor $T^{ab}{}_c$ can be contracted over its 2nd and 3rd indices to give the vector $S^a = T^{ab}{}_b$; this is shown below to be a vector:

$$S'^a = T'^{ab}{}_b = \frac{\partial x'^a}{\partial x^c} \frac{\partial x'^b}{\partial x^d} \frac{\partial x^e}{\partial x'^b} T^{cd}{}_e = \frac{\partial x'^a}{\partial x^c} \delta^e_d T^{cd}{}_e = \frac{\partial x'^a}{\partial x^c} T^{cd}{}_d = \frac{\partial x'^a}{\partial x^c} S^c$$

Contraction converts a type-(k, l) tensor into a type-$(k - 1, l - 1)$ tensor.

3.4.2 Quotient Theorem

The quotient theorem is a way of testing whether an object is a tensor or not: if one contracts this object with an arbitrary tensor, and the result is another tensor, then the object in question must have also been a tensor.

[2] This is the same relationship that bras have with ~~panties~~ kets in quantum mechanics.

[3] Not all rank 2 tensors can be constructed as the outer product of two vectors, but all rank 2 tensors *can* be constructed from a *linear combination of* outer products of vectors.

3.5 Tensor Symmetry

A tensor is described as symmetric if, for example $S_{ab} = S_{ba}$. A tensor is *antisymmetric* if $A_{ab} = -A_{ba}$. The properties of symmetry and antisymmetry can be extended to higher-order tensors. For instance, the rank-3 tensor $\mathbf{U}$, with $U^{ab}{}_c = -U^{ba}{}_c$, is said to be *antisymmetric in its first and second indices*. These properties are of the tensor itself, being preserved between coordinate systems.

A type-(0,2) or type-(2,0) tensor can always be written as the sum of a symmetric tensor and an antisymmetric tensor:

$$T_{ab} = \frac{1}{2}(T_{ab} + T_{ba}) + \frac{1}{2}(T_{ab} - T_{ba})$$

The first term is symmetric and denoted $T_{(ab)}$; the second is antisymmetric and denoted $T_{[ab]}$. Extending this,

$$T_{(ab)c} = \frac{1}{2}(T_{abc} + T_{bac}) \qquad T_{[abc]} = \frac{1}{6}(T_{abc} - T_{acb} + T_{bca} - T_{bac} + T_{cab} - T_{cba})$$

where the prefactors ensure that symmetrising an already-symmetric tensor does nothing.

3.6 Metric Tensor

$$g'_{cd} = \frac{\partial x^a}{\partial x'^c}\frac{\partial x^b}{\partial x'^d} g_{ab}$$

The metric's components transform like those of a type-(0,2) tensor, $\mathbf{g}$. If for a moment we step back into the formal view that such tensors are maps from two copies of $T_P(\mathcal{M})$ to $\mathbb{R}$, we define the inner product between two vectors with components u^a and v^a as the result of mapping these vectors to $\mathbb{R}$ using $\mathbf{g}$: the inner product is the scalar $g_{ab}u^a v^b$.

It was mentioned earlier that the dual vector space $T^*_P(\mathcal{M})$ is the space of linear maps from $T_P(\mathcal{M})$ to $\mathbb{R}$. It turns out that there is an *isomorphism* between the two spaces, that is, for every vector in $T_P(\mathcal{M})$ there is exactly one dual vector in $T^*_P(\mathcal{M})$. The most useful way to pair the two up is to define the dual vector $\boxed{v_a \equiv g_{ab}v^b}$. In this way any vector in T_P is assigned a dual vector in T^*_P by "lowering its index" using g_{ab}. The square-norm of the vector is then $v_a v^a$, which is nice. On a Riemannian manifold, this is at least 0, and only equal to 0 if all the v^a (and hence v_a) are 0. On a pseudo-Riemannian manifold, this is no longer the case and the square-norm can be negative. The "length" of a vector is

$$\boxed{|\mathbf{v}| = \left|g_{ab}v^a v^b\right|^{1/2} = |v_a v^a|^{1/2}} > 0$$

We extend the lowering ability of g_{ab} and allow it to create new objects out of any tensor with an upstairs index, for example $g_{ad}g_{be}T^{ab}{}_c = T_{dec}$.

Consider the inverse of g_{ab}, written $(g^{-1})^{ab}$:

$$(g^{-1})^{ab} g_{bc} = \delta^a_c$$

Transforming this, using the fact that δ^a_c is an isotropic tensor whose components are invariant of coordinate system, we find that g^{-1} transforms as a type-(2,0) tensor. In the same way that we use the same letter v for v_a and v^a above, we will subsequently write $(g^{-1})^{ab}$ as g^{ab} for convenience:

$$\boxed{g^{ab}g_{bc} = \delta^a_c}$$

Furthermore, this notation is consistent with the lowering property of g_{ab}:

$$g_{ab}g_{cd}g^{bd} = g_{ab}\delta^c_b = g_{ac}$$

Also, using the raising property and multiplying by g^{ab}:

$$g_{ab}v^b = v_a \Rightarrow g^{ca}g_{ab}v^b = g^{ca}v_a \Rightarrow \boxed{v^c = g^{ca}v_a}$$

so g^{ab} has the ability to *raise* the indices of a dual vector to a vector.

We can now do all sorts of raising and lowering operations on tensors:

$$g_{ab}g^{cd}T^b{}_{de} = T_a{}^c{}_e$$

Note we are being very careful to keep the ordering of the indices clear, as for example the quantity $T^a{}_c = g^{ab}T_{bc}$ is generally different to $T_c{}^a = g^{ab}T_{cb}$, unless $T_{ab} = T_{ba}$.

4 Vector & Tensor Calculus

Remember that *a useful derivative of a tensor must also be a tensor.*

4.1 Covariant Derivative

If we try to construct the derivative of a vector v^b with respect to some (general) coordinate x^a, we obtain an object which is unfortunately not a tensor:

$$\frac{\partial v'^b}{\partial x'^a} = \frac{\partial x^c}{\partial x'^a}\frac{\partial}{\partial x^c}\left(\frac{\partial x'^b}{\partial x^d}v^d\right) = \frac{\partial x^c}{\partial x'^a}\frac{\partial x'^b}{\partial x^d}\frac{\partial v^d}{\partial x^c} + \frac{\partial x^c}{\partial x'^a}\frac{\partial^2 x'^b}{\partial x^c \partial x^d}v^d$$

The first term is exactly what we want, but the second term, due to the curvature of the manifold is unwelcome. We define the covariant derivative as:

$$\boxed{\nabla_a v^b = \frac{\partial v^b}{\partial x^a} + \Gamma^b_{ac}v^c}$$

where the coefficients Γ^b_{ac} (the *connection coefficients*) ensure this transforms as a tensor. We can also define a *contravariant derivative* by raising the index on the operator:

$$\nabla^a v^b = g^{ac}\nabla_c v^b$$

The connection coefficients account for the fact that the tangent space changes from point to point. To find what they are, we use the assumption that the covariant derivative does indeed transform as a tensor, and substitute the definition above:

$$\nabla'_a v'^b = \frac{\partial x^c}{\partial x'^a}\frac{\partial x'^b}{\partial x^d}\nabla_c v^d \qquad \Rightarrow \qquad \Gamma'^b_{af} = \frac{\partial x'^b}{\partial x^e}\frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^f}\Gamma^e_{cd} - \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^f}\frac{\partial^2 x'^b}{\partial x^c \partial x^d}$$

where the final term excludes the Γ coefficients from tensorhood. Any coefficients which satisfy the above form a viable connection, but the *Levi-Civita connection* is the most natural. This connection relies on six reasonable axioms:

- For a scalar field ϕ, the covariant derivative is the same as the regular derivative:

$$\nabla_a \phi = \frac{\partial \phi}{\partial x^a}$$

  This seems reasonable because ϕ does not make reference to any basis vectors, which change from place to place, so just the regular derivative will do.

- The covariant derivative is linear

- The covariant derivative commutes with contraction over indices

- The covariant derivative obeys the Leibniz differentiation rule, for example

$$\nabla_a \big(T^{bc} S^{de}\big) = \nabla_a \big(T^{bc}\big) S^{de} + T^{bc} \nabla_a \big(S^{de}\big)$$

- "Metric compatibility": $\boxed{\nabla_a g_{bc} = 0}$.

- The connection is "torsion-free", that is, for any ϕ, $\nabla_{[a} \nabla_{b]} \phi \equiv \frac{1}{2}(\nabla_a \nabla_b \phi - \nabla_b \nabla_a \phi) = 0$

4.1.1 Covariant Derivative of a Dual Vector

$X_a v^a$ is a scalar, so using the Leibniz rule:

$$\nabla_a \big(X_b v^b\big) = \frac{\partial (X_b v^b)}{\partial x^a} \qquad \Rightarrow \qquad \boxed{\nabla_a X_b = \frac{\partial X_b}{\partial x^a} - \Gamma^c_{ab} X_c}$$

Γ is symmetric in its lower indices. To show this, consider:

$$\nabla_a \nabla_b \phi = \nabla_a \partial_b \phi = \partial_a \partial_b \phi - \Gamma^c_{ab} \partial_c \phi \qquad \Rightarrow \qquad \nabla_{[a} \nabla_{b]} \phi = -\Gamma^c_{[ab]} \partial_c \phi$$

For this to be equal to 0 as postulated, Γ must be symmetric in its lower indices.

4.1.2 Covariant Derivative of a Rank 2 Tensor

All the operators involved are linear, so wlog we can consider the covariant derivative of a tensor $T^{ab} = u^a v^b$. Using the Leibniz rule:

$$\begin{aligned}
\nabla_a T^{bc} = \nabla_a u^b v^c + u^b \nabla_a v^c &= \partial_a u^b v^c + \Gamma^b_{ad} u^d v^c + u^b \partial_a v^c + u^b \Gamma^c_{ad} v^d \\
&= \partial_a (u^b v^c) + \Gamma^b_{ad} u^d v^c + \Gamma^c_{ad} u^b v^d \\
&= \partial_a T^{bc} + \Gamma^b_{ad} T^{dc} + \Gamma^c_{ad} T^{bd}
\end{aligned}$$

Similarly, we may derive:

$$\nabla_a T^b{}_c = \partial_a T^b{}_c + \Gamma^b_{ad} T^d{}_c - \Gamma^d_{ac} T^b{}_d \qquad\qquad \nabla_a T_{bc} = \partial_a T_{bc} - \Gamma^d_{ab} T_{dc} - \Gamma^d_{ac} T_{bd}$$

For every upstairs index, the covariant derivative obtains a $+\Gamma$ term, and for every downstairs index the formula has a $-\Gamma$ term. There is always only one sensible possibility for where to put all the indices, so just get the signs right.

4.1.3 The Levi-Civita Connection / Christoffel Symbols

Metric compatibility gives:

$$\partial_a g_{bc} - \Gamma^d_{ab} g_{bd} - \Gamma^d_{ab} g_{cd} = 0$$

Cyclically permuting the indices, and subtracting one from the other two using the symmetry of Γ, we find

$$\boxed{\Gamma^c_{ab} = \frac{1}{2} g^{cd} (\partial_a g_{cd} + \partial_b g_{ad} - \partial_d g_{ab})}$$

To get the indices in the right place:

1. A factor of $1/2$, a g^{ab} factor, then three $\partial_a g_{bc}$ terms, with two $+$ and one $-$.
2. There is exactly one dummy index. With the three free indices in the Γ you are trying to work out, this gives four indices in total.
3. The two upstairs indices of the g^{ab} factor can only be the upstairs index of the Γ and the dummy index, as both the other indices are free and downstairs so they can't be the indices of the inverse metric, which are upstairs.
4. The three $\partial_\Box$ terms have three lower indices to choose from: the lower indices of the Γ, and the dummy index. The terms with a $+$ sign are those where the metric is differentiated with respect to a free index of the Γ; the term with a $-$ sign is that where the metric is differentiated with respect to the dummy index.

For the special (but common) case where the metric is diagonal, the Christoffel symbols simplify. For $a \neq b \neq c$ and suspending the summation convention,

$$\boxed{\Gamma^a_{bc} = 0} \quad \boxed{\Gamma^a_{bb} = -\frac{1}{2g_{aa}} \partial_a g_{bb}} \quad \boxed{\Gamma^a_{aa} = \frac{1}{2g_{aa}} \partial_a g_{aa}} \quad \boxed{\Gamma^a_{ab} = \frac{1}{2g_{aa}} \partial_b g_{aa}}$$

noting the minus sign in the second term.

The quantity Γ^a_{ab}: beginning from metric compatibility and multiplying by g^{bc}, one finds:

$$\Gamma^b_{ba} == \frac{1}{2} |g|^{-1} \partial_a |g| = |g|^{-1/2} \partial_a |g|^{1/2}$$

where we have used $M^{bc} \partial_a M_{bc} = \mathrm{Tr}\,(M^{-1} \partial_a M) = |M|^{-1} \partial_a |M|$ for an invertible matrix M; g is the determinant of the matrix g_{ab}.

4.1.4 Divergence and Curl

The *divergence* of a vector field v^a is $\nabla_a v^a$, which can be found to be:

$$\nabla_a v^a = |g|^{-1/2} \partial_a \big(|g|^{1/2} v^a \big)$$

The familiar definition of the *curl* of a vector field doesn't really work. Instead, define the *curl tensor* of a *dual* vector field X_a as the tensor $\nabla_a X_b - \nabla_b X_a$ which turns out to be independent of the connection. We also recover the familiar property that the curl of a gradient of a scalar field is 0, as $\nabla_a \nabla_b \phi - \nabla_b \nabla_a \phi \equiv 0$ axiomatically.

The Laplacian operator is $\nabla^2 = \nabla_a \nabla^a = \nabla_a \big(g^{ab} \nabla_b \big) = g^{ab} \nabla_a \nabla_b$:

$$\nabla^2 \phi = |g|^{-1/2} \partial_a \big(|g|^{1/2} g^{ab} \partial_b \phi \big)$$

4.2 Intrinsic Derivative

The intrinsic derivative is the derivative of a tensor along some parametrised curve $x^a = x^a(u)$.

$$\frac{Dv^a}{Du} \equiv \frac{\mathrm{d}x^b}{\mathrm{d}u}\boldsymbol{\nabla}_b v^a = \frac{\mathrm{d}x^b}{\mathrm{d}u}(\partial_b v^a + \Gamma^a_{bc}v^c) \qquad \Rightarrow \qquad \boxed{\frac{Dv^a}{Du} = \frac{\mathrm{d}v^a}{\mathrm{d}u} + \frac{\mathrm{d}x^b}{\mathrm{d}u}\Gamma^a_{bc}v^c}$$

contracting the covariant derivative with the tangent vector $\mathrm{d}x^b/\mathrm{d}u$. Being a contraction, the intrinsic derivative is itself a tensor, whereas $\mathrm{d}v^a/\mathrm{d}u$ by itself is not; what we would intuitively want the derivative to be is not a tensor, so we need a connection term. The expression involves only $v^a(u)$, so there is no need for v^a to be defined off the path $x^a = x^a(u)$ to calculate the intrinsic derivative.

This derivative can also be calculated for higher-order tensors:

$$\frac{DT^a{}_b}{Du} = \frac{\mathrm{d}x^c}{\mathrm{d}u}\boldsymbol{\nabla}_c T^a{}_b = \frac{\mathrm{d}T^a{}_b}{\mathrm{d}u} + \frac{\mathrm{d}x^c}{\mathrm{d}u}\Gamma^a_{cd}T^d{}_b - \frac{\mathrm{d}x^c}{\mathrm{d}u}\Gamma^d_{cb}T^a{}_d$$

Finally, any intrinsic derivative of the metric tensor is $Dg_{ab}/Du = \boldsymbol{\nabla}_c g_{ab}\,\mathrm{d}x^c/\mathrm{d}u = 0$

4.2.1 Parallel Transport

Suppose we have a vector v^a defined at a point, and a curve $x^a(u)$, and we wish to define a vector field along that curve so that the vector has the same magnitude and "direction" at all points along it. This is described as "parallel transporting" the vector, and is satisfied by imposing $\boxed{Dv^a/Du = 0}$. Again, we would intuitively expect it to be $\mathrm{d}v^a/\mathrm{d}u$ (the components each staying the same along the path), but we need a connection term to account for manifold curvature. $Dv^a/Du = 0$ is an ODE for the v^a, integration constants being specified by knowing the values of the vector's components at the starting point.

Dot products of parallel-transported vectors are preserved under parallel transport, as:

$$\frac{D(\mathbf{u}\cdot\mathbf{v})}{Du} = \frac{D}{Du}\big(g_{ab}u^a v^b\big) = \frac{Dg_{ab}}{Du}u^a v^b + g_{ab}\left(\frac{Du^a}{Du}v^b + u^a\frac{Dv^b}{Du}\right) = 0$$

because the first term involves the intrinsic derivative of the metric and the second two involve terms like Du^a/Du which by supposition are 0.

4.2.2 Geodesics

Geodesics are curves defined by either:

- A curve along which the tangent vector parametrised by path length ($t^a = \mathrm{d}x^a/\mathrm{d}s$) is parallel transported
- A curve of extremal distance between two points

Taking the first definition above, the defining equation becomes:

$$0 = \frac{Dt^a}{Ds} = \frac{\mathrm{d}x^b}{\mathrm{d}s}\boldsymbol{\nabla}_b t^a = \frac{\mathrm{d}x^b}{\mathrm{d}s}\left[\frac{\partial t^a}{\partial x^b} + \Gamma^a_{bc}t^c\right] = \frac{\mathrm{d}t^a}{\mathrm{d}s} + \Gamma^a_{bc}\frac{\mathrm{d}x^b}{\mathrm{d}s}t^c = \frac{\mathrm{d}^2x^a}{\mathrm{d}s^2} + \Gamma^a_{bc}\frac{\mathrm{d}x^b}{\mathrm{d}s}\frac{\mathrm{d}x^c}{\mathrm{d}s}$$

$$\boxed{\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0}$$

It can be seen that if the curve is not parametrised by s, but by some other parameter u such that $\mathrm{d}u/\mathrm{d}s$ is constant (*affine* parameters), this constant would cancel and the same result would be found.

The second definition leads to the same result, and also shows how non-affine parameters are accounted for. The length of the curve is $\int_A^B \mathrm{d}s = \int_{u_A}^{u_B} |g_{ab}\dot{x}^a\dot{x}^b|^{1/2}\,\mathrm{d}u$, where $\dot{x}^a \equiv \mathrm{d}x^a/\mathrm{d}u$. Calling the integrand $F = \mathrm{d}s/\mathrm{d}u = \dot{s}$ and applying the Euler-Lagrange equation gives a related form

$$\frac{\partial F}{\partial x^c} = \frac{\mathrm{d}}{\mathrm{d}u}\frac{\partial F}{\partial \dot{x}^c} \qquad \Rightarrow \qquad \ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = \frac{\ddot{s}}{\dot{s}}\dot{x}^a$$

Thus if an affine parameter is used (giving $\ddot{s} = 0$), the RHS is 0 and the same equation as above is recovered.

Much of the above does not apply to null curves, for which $s = 0$ along the curve, so it cannot be used as a parameter. In this case, we simply use whatever parameter u is available, forming $t^a = \mathrm{d}x^a/\mathrm{d}u$ and setting $Dt^a/Du = 0$. This still yields a unique curve, a *null geodesic.*

The definition of a geodesic can also be derived from a Lagrangian. As we are here looking to extremise path length (or, equally, path length squared), we have $\mathcal{L} = g_{ab}\dot{x}^a\dot{x}^b$. Because $\mathcal{L}$ does not explicitly depend on u, we can take the first integral to find that the quantity

$$\dot{x}^c\frac{\partial\mathcal{L}}{\partial\dot{x}^c} - \mathcal{L} = g_{ac}\dot{x}^a\dot{x}^c = \mathcal{L}$$

is conserved along the path. Further, if $\mathcal{L}$ does not depend on some “ignorable” coordinate x^i, then the conjugate momentum $\partial\mathcal{L}/\partial\dot{x}^i$ is also conserved along the curve.

4.3 Curvature

The Riemann curvature tensor $R_{abc}{}^d$ is defined by:

$$\boxed{(\boldsymbol{\nabla}_a\boldsymbol{\nabla}_b - \boldsymbol{\nabla}_b\boldsymbol{\nabla}_a)v_c = R_{abc}{}^d v_d}$$

To work out its components, we first have:

$$\begin{aligned}
\boldsymbol{\nabla}_a\boldsymbol{\nabla}_b v_c &= \boldsymbol{\nabla}_a\big(\partial_b v_c - \Gamma^d_{bc}v_d\big) = \partial_a\big(\partial_b v_c - \Gamma^d_{bc}v_d\big) - \Gamma^e_{ab}\big(\partial_e v_c - \Gamma^d_{ec}v_d\big) - \Gamma^e_{ac}\big(\partial_b v_e - \Gamma^d_{be}v_d\big)\\
\Rightarrow R_{abc}{}^d v_d &= -\partial_a\big(\Gamma^d_{bc}v_d\big) + \partial_b\big(\Gamma^d_{ac}v_d\big) - \Gamma^d_{ac}\partial_b v_d + \Gamma^d_{bc}\partial_a v_d + (\Gamma^e_{ac}\Gamma^d_{be} - \Gamma^e_{bc}\Gamma^d_{ae})v_d\\
&= (-\partial_a\Gamma^d_{bc} + \partial_b\Gamma^d_{ac} + \Gamma^e_{ac}\Gamma^d_{be} - \Gamma^e_{bc}\Gamma^d_{ae})v_d\\
&\Rightarrow \boxed{R_{abc}{}^d = -\partial_a\Gamma^d_{bc} + \partial_b\Gamma^d_{ac} + \Gamma^e_{ac}\Gamma^d_{be} - \Gamma^e_{bc}\Gamma^d_{ae}}
\end{aligned}$$

If a manifold is globally flat, $\exists$ coordinates in which g_{ab} takes a constant diagonal form, and so $\Gamma = 0$ everywhere and $R_{abc}{}^d = 0$. In local Cartesian coordinates on a general manifold (a family of which always exists), g_{ab} takes a constant diagonal form only locally, the $\partial_a g_{bc}$ are locally 0, so $\Gamma = 0$ locally, but $\partial_a\partial_b g_{cd} \neq 0$ so $\partial\Gamma \neq 0$.

Memory aids: there are two terms involving $\partial\Gamma$ and two involving $\Gamma\Gamma$; in each term there is a Γ with either $\Gamma^\square_{bc}$ or $\Gamma^\square_{ac}$. Terms with $\Gamma^\square_{bc}$ have $-$; those with $\Gamma^\square_{ac}$ have $+$.

The Riemann tensor has several symmetries, many of which are easiest to derive in local Cartesian coordinates in which $\Gamma = 0$. They are summarised by:

$$\boxed{R_{abcd} = -R_{bacd} = -R_{abdc} = R_{cdab}} \qquad \boxed{R_{abcd} + R_{bcad} + R_{cabd} = 0}$$

It turns out that these symmetries leave $\frac{1}{12}N^2(N+1)(N-1)$ independent components of the tensor in N dimensions; $= 20$ for $N = 4$.

A useful property of $R_{abc}{}^d$, also best derived in local Cartesians, is the Bianchi identity:

$$\boxed{\boldsymbol{\nabla}_a R_{bcd}{}^e + \boldsymbol{\nabla}_b R_{cad}{}^e + \boldsymbol{\nabla}_c R_{abd}{}^e = 0} \qquad \Rightarrow \qquad \boldsymbol{\nabla}_{[a} R_{bc]d}{}^e = 0 \qquad \text{(Bianchi)}$$

4.3.1 Ricci Tensor and Scalar

Contracting $R_{abc}{}^d$ once gives the Ricci tensor $\boxed{R_{ab} \equiv R_{cab}{}^c = R_{ba}}$. It is possible for a manifold to not be flat ($R_{abc}{}^d = 0$), yet $R_{ab} = 0$. Contracting again gives the Ricci scalar $\boxed{R \equiv g^{ab} R_{ab}}$.

Contracting over the Bianchi identity over (be) and then (ad) gives:

$$\boldsymbol{\nabla}^a \left(R_{ab} - \frac{1}{2} g_{ab} R \right) = 0$$

That is, the divergence of the *Einstein tensor*, $G_{ab} \equiv R_{ab} - \frac{1}{2} g_{ab} R$, is 0.

4.3.2 Consequences of Curvature

Parallel Transport around a Closed Curve. Parallel transport gives:

$$\frac{Dv^a}{Du} = 0 \qquad \Rightarrow \qquad \frac{\mathrm{d}v^a}{\mathrm{d}u} = -\Gamma^a_{bc} \frac{\mathrm{d}x^b}{\mathrm{d}u} v^c$$

Integrating the RHS around a closed loop will in general be non-zero giving a change in the vector component; this can be done analytically in the limit of small curves. The ingredients are, for a curve starting at the origin parametrised by u with $u = 0$ at the origin:

$$\Gamma^a_{bc}(u) = \Gamma^a_{bc}(0) + \partial_d \Gamma^a_{bc}\Big|_0 x^d(u)$$
$$v^c(u) = v^c(0) + \partial_e v^c\Big|_0 x^e(u) = v^c(0) - \Gamma^c_{ef}(0) x^e(u) v^f(0)$$

Integrating the RHS to first order in x^a then gives

$$\Delta v^a = \frac{1}{2} R_{bcd}{}^a(0) v^d(0) \oint x^b \, \mathrm{d}x^c$$

And so the amount by which a vector changes on being transported around a small closed loop depends on the curvature of the manifold.

Geodesic Deviation. Two infinitesimally-separated geodesics $x^a(u)$ and $y^a(u)$ have equations:

$$\ddot{x}^a + \Gamma^a_{bc}(x)\dot{x}^b\dot{x}^c = 0 \qquad\qquad \ddot{y}^a + \Gamma^a_{bc}(y)\dot{y}^b\dot{y}^c$$

Let $\xi = x - y$. By writing out $D^2\xi^a/Du^2$, and comparing with the difference of the two above equations, we obtain:

$$\boxed{\frac{D^2\xi^a}{Du^2} - R_{dbc}{}^a\dot{x}^b\dot{x}^c\xi^d = 0}$$

If the manifold is flat, $\mathbf{R} = \mathbf{0}$ and a Cartesian coordinate system exists so $D^2\xi^a/Du^2 = \partial^2\xi^a/\partial u^2$, so we get $\partial^2\xi^a/\partial u^2 = 0$, as is the flat result. If the manifold is not flat, these will diverge or converge, as meridians on the surface of the Earth.

5 Minkowski Spacetime

Minkowski spacetime is the manifold on which Special Relativity happens. It has metric

$$g_{\mu\nu} = \eta_{\mu\nu} \equiv \mathfrak{diag}(1, -1, -1, -1)$$

Clearly the inverse metric, $\eta^{\mu\nu}$, has the same components. All the components are constant, so all of the connection terms are 0 and most of the previous section can briefly be forgotten. This also means that one can describe positions on the entire manifold using four Euclidean coordinates (ct, x, y, z). If we lower the index of a vector v^μ using $\eta_{\mu\nu}$, the time component will stay the same and the spatial components will all be reversed.

For a 4-dimensional manifold, there are 6 degrees of freedom when converting from one completely general initial coordinate system to a new one with a constant diagonal form. If the initial metric (in $\mathcal{S}$, say) is in fact $\eta_{\mu\nu}$, this still applies; we can transform to a new coordinate system ($\mathcal{S}'$) where the new metric is *also* $\eta_{\mu\nu}$, and this transformation, (in this case the Lorentz transformation) has 6 degrees of freedom:

$$g'_{\mu\nu} = \eta_{\mu\nu} = \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu}\eta_{\rho\sigma} = g_{\mu\nu} \qquad \text{and conversely} \qquad \eta_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\eta_{\rho\sigma}$$

It can be shown that these constraints restrict the partial derivatives $\partial x'^\mu/\partial x^\nu$ to be constants, written $\Lambda^\mu{}_\nu$ and given by:

$$[\Lambda^\mu{}_\nu] = \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

for a transformation which is a linear boost along the x-axis. Thus $\eta_{\mu\nu} = \Lambda^\rho{}_\mu\Lambda^\sigma{}_\nu\eta_{\rho\sigma}$. Rearranging this, the *inverse* Lorentz transformations are therefore:

$$(\Lambda^{-1})^\rho{}_\mu = \eta_{\rho\sigma}\eta^{\mu\nu}\Lambda^\sigma{}_\nu$$

suggesting that maybe we should write $(\Lambda^{-1})^\rho{}_\mu$ as $\Lambda_\sigma{}^\rho$ and be very careful about where we put indices. The Λ do **not** form the components of a tensor; this is just a funky shorthand.

The magnitudes of vectors are preserved under Lorentz transformations:

$$|\mathbf{v}'|^2 = \eta_{\mu\nu}v'^\mu v'^\nu = \eta_{\mu\nu}\Lambda^\mu{}_\rho v^\rho\Lambda^\nu{}_\sigma v^\sigma = \eta_{\rho\sigma}v^\rho v^\sigma = |\mathbf{v}|^2$$

Because all the connection terms of Minkowski space are 0, the operators $\boldsymbol{\nabla}_a$ and D/Du are here equal to ∂_a and $\mathrm{d}/\mathrm{d}u$ respectively. We will often use the latters when talking about dynamics in Minkowski space (such as velocity being the time derivative of position) where there is no distinction, but when we do dynamics on more general manifolds, with non-zero connections, these familiar operators will be replaced by the connection-inclusive $\boldsymbol{\nabla}_a$ and D/Du.

5.1 Dynamical 4-Vectors

Vectors in Minkowski space are often called *4-vectors*.

5.1.1 Position $\mathbf{x}$

$\mathbf{x}$ has components $x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z) = (ct, \boldsymbol{x})$, where $\boldsymbol{x}$ is the 3D position vector. The trajectory of a particle as it moves through space and time is a curve in Minkowski space. For massive particles, these curves can be parametrised by the proper time τ, the time recorded by a particle in its IRF $\mathcal{F}$. From one moment to another in $\mathcal{F}$, the particle does not move by definition, so $\mathrm{d}s^2 = c^2\,\mathrm{d}\tau^2 \Rightarrow \mathrm{d}s = c\,\mathrm{d}\tau$. The proper time is thus an affine parameter, as the spacetime distance s along the path is linearly dependent on it.

5.1.2 Velocity $\mathbf{u} \equiv \mathrm{d}\mathbf{x}/\mathrm{d}\tau$

$\mathbf{u}$ is a tangent to the curve the particle follows. For a massive particle, this vector points within the light cone at each point on the curve, and is therefore timelike. As such, the magnitude of the velocity vector is positive; it is also in fact positive for any metric:

$$|\mathbf{u}|^2 = g_{\mu\nu}u^\mu u^\nu = g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = \frac{g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu}{\mathrm{d}\tau^2} = \left(\frac{\mathrm{d}s}{\mathrm{d}\tau}\right)^2 = c^2$$

Thus $|\mathbf{u}| = c$ on any manifold, including Minkowski spacetime. Differentiating the components:

$$\mathbf{u} = \frac{\mathrm{d}}{\mathrm{d}\tau}(ct, x, y, z) = \frac{\mathrm{d}t}{\mathrm{d}\tau}\left(c, \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t}\right) = \frac{\mathrm{d}t}{\mathrm{d}\tau}(c, \boldsymbol{u})$$

where $\boldsymbol{u}$ is the regular 3D velocity $(\mathrm{d}x/\mathrm{d}t\ ,\ \mathrm{d}y/\mathrm{d}t\ ,\ \mathrm{d}z/\mathrm{d}t)$. The magnitude of this vector must be c as shown above, so we have:

$$\frac{\mathrm{d}t}{\mathrm{d}\tau}\sqrt{c^2 - \boldsymbol{u}^2} = c \qquad \Rightarrow \qquad \frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma$$

as before. Thus $\mathbf{u} = \gamma(c, \boldsymbol{u})$

By using the Lorentz transformations $u'^\mu = \Lambda^\mu{}_\nu u^\nu$, one can derive the velocity transformation laws found earlier. The equality of the first component relates $\gamma_{u'}$ of the new frame to γ_u of the old frame in terms of γ_v where v is the relative velocity of the frames. This can then be used to deduce the transformed velocities in the x, y and z directions.

5.1.3 Momentum $\mathbf{p} \equiv m\mathbf{u}$

For massive particles $\mathbf{p}$ has components $m\mathbf{u} = (\gamma mc, \gamma m\boldsymbol{u})$, and so is also a timelike vector. It can be shown that relativistic interacting particles do not conserve the 3-vector $\Sigma_i m_i \boldsymbol{u}_i$ but *do* conserve $\Sigma_i \gamma_{u_i} m_i \boldsymbol{u}_i$, so we identify $\boldsymbol{p} = \gamma m \boldsymbol{u}$. Similarly, the total energy can be shown to be $E = \gamma m c^2$, so we have:

$$\mathsf{p} = \left(\frac{E}{c}, \boldsymbol{p}\right)$$

The square magnitude of the 4-velocity is c^2, so that of the 4-momentum is $m^2 c^2$, and

$$\boxed{E^2 - \boldsymbol{p}^2 c^2 = m^2 c^4}$$

The LHS is called the *energy-momentum invariant*, as it is unchanged under a Lorentz transformation (though E and $\boldsymbol{p}$ will of course individually change).

5.1.4 Acceleration $\mathsf{a} \equiv \mathrm{d}\mathsf{u}/\mathrm{d}\tau$

To perform the derivative and find its components in terms of $\boldsymbol{a}$, we first need the important formula:

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(1 - \frac{\boldsymbol{u}^2}{c^2}\right)^{-1/2} = -\frac{1}{2}\gamma^3 \frac{-2\boldsymbol{u}}{c^2} \cdot \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = \frac{\gamma^3}{c^2}\boldsymbol{u} \cdot \boldsymbol{a}$$

Hence:

$$\mathsf{a} = \frac{\mathrm{d}\mathsf{u}}{\mathrm{d}t} = \gamma \frac{\mathrm{d}}{\mathrm{d}\tau}(\gamma c, \gamma \boldsymbol{u}) = \left(\frac{\gamma^4}{c}\boldsymbol{u} \cdot \boldsymbol{a}, \frac{\gamma^4}{c^2}(\boldsymbol{u} \cdot \boldsymbol{a})\boldsymbol{u} + \gamma^2 \boldsymbol{a}\right) = \gamma^2 \left(\frac{\gamma^2}{c}\boldsymbol{u} \cdot \boldsymbol{a}, \frac{\gamma^2}{c^2}(\boldsymbol{u} \cdot \boldsymbol{a})\boldsymbol{u} + \boldsymbol{a}\right)$$

a must be orthogonal to u, as:

$$\eta_{\mu\nu} u^\mu a^\nu = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}(\eta_{\mu\nu} u^\mu u^\nu) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau} c^2 = 0$$

The length of a 4-vector is invariant under a Lorentz transformation. As such, we can find the length of a by choosing the frame in which $\boldsymbol{u} = \boldsymbol{0}$, that is, $\mathcal{F}$: in this frame, $\mathsf{a} = (0, \boldsymbol{a}_{\mathcal{F}})$, and so $|\mathsf{a}|^2 = -\boldsymbol{a}_{\mathcal{F}}^2$. Therefore a is a spacelike vector.

5.1.5 Force $\mathsf{f} = m\mathsf{a}$

Force. The force 4-vector (or "fource" if you will), f, is given by $m\mathsf{a}$, and thus is also spacelike and orthogonal to u. Alternatively, one could define it by $\mathsf{f} = \mathrm{d}\mathsf{p}/\mathrm{d}\tau$, in which case its components would be found as:

$$\mathsf{f} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{E}{c}, \boldsymbol{p}\right) = \gamma \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{E}{c}, \boldsymbol{p}\right) = \gamma\left(\frac{\boldsymbol{f} \cdot \boldsymbol{u}}{c}, \boldsymbol{f}\right)$$

where $\boldsymbol{f}$ is the 3-force, and $\boldsymbol{f} \cdot \boldsymbol{u} = \mathrm{d}E/\mathrm{d}t$.

5.2 Light

Momentum and energy are still conserved in relativistic interactions (provided one is careful to include factors of γ), even those involving photons. As such we must still be able to give photons 4-momentum to include their contributions. It is still given by $\mathsf{p} = (E/c, \boldsymbol{p})$, and still has a magnitude of $m^2 c^2$, but photons have $m = 0$ so their 4-momentum must be *null* and we must have $E^2 - \boldsymbol{p}^2 c^2 = 0$, as is familiar with light.

The proper time τ cannot be used as a parameter for the paths photons travel along as $\tau = 0$ for the entire curve. We therefore cannot really define a 4-velocity $\mathbf{u}$ for a photon, as this is defined in terms of a derivative with respect to τ, though we can define their $\mathbf{p}$ as above. It can also be defined $\mathbf{p} = \mathrm{d}\mathbf{x}/\mathrm{d}\zeta$ where ζ is any parameter for the null curve. It doesn't matter what the parameter is – the parameter does not affect the direction, only the magnitude, and the magnitude is always 0.

5.2.1 Wavevector $\mathbf{k} \equiv \mathbf{p}/\hbar$

As $\mathbf{p}$ is null, so must be $\mathbf{k}$ and we thus have:

$$\mathbf{k} = (|\boldsymbol{k}|, \boldsymbol{k}) = \left(\frac{2\pi}{\lambda}, \boldsymbol{k}\right)$$

As the 4-wavevector transforms like any 4-vector under a Lorentz transformation, we can use the first component to rederive the Doppler Effect. If $\boldsymbol{k}$ is just in the x direction, we have:

$$\frac{2\pi}{\lambda'} = \gamma\frac{2\pi}{\lambda} - \gamma\beta\frac{2\pi}{\lambda} = \gamma(1-\beta)\frac{2\pi}{\lambda} = \sqrt{\frac{1-\beta}{1+\beta}}\frac{2\pi}{\lambda} \qquad \Rightarrow \qquad f' = \sqrt{\frac{1-\beta}{1+\beta}}f \qquad \text{(as before)}$$

5.3 Maxwell's Equations

Given that light was what got us into this mess in the first place, it is possible to express Maxwell's Equations (below) in a relativistic framework – that is, as tensor equations.

$$\boxed{\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \cdot \boldsymbol{B} = 0 \qquad \nabla \times \boldsymbol{E} = -\dot{\boldsymbol{B}} \qquad \nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0\epsilon_0\dot{\boldsymbol{E}}}$$

5.3.1 The Faraday Tensor $\mathbf{F}$

The Lorentz force, $\boldsymbol{f} = q(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B})$, depends linearly on the velocity, and is proportional to charge (a scalar), suggesting that the *fource* should too. We introduce the *Faraday Tensor* $\mathbf{F}$, so that the fource is given by

$$f^\mu = qF^\mu{}_\nu u^\nu \qquad \Rightarrow \qquad f_\mu = qF_{\mu\nu}u^\nu$$

For the fource and velocity to be orthogonal as shown earlier, $\mathbf{F}$ is required to be antisymmetric:

$$0 = f_\mu u^\mu = qF_{\mu\nu}u^\mu u^\nu = \frac{1}{2}q(F_{\mu\nu} + F_{\nu\mu})u^\mu u^\nu$$

In Minkowski space we found that the expression for the fource is

$$f^\mu = \gamma\left(\frac{\boldsymbol{f}\cdot\boldsymbol{u}}{c}, \boldsymbol{f}\right) \qquad \Rightarrow \qquad f_\mu = \left(\gamma\frac{\boldsymbol{f}_i\mathbf{u}_i}{c}, -\boldsymbol{f}_j\right)$$

where we allow Latin indices like i to run from 1 to 3, and note that when we are dealing with Cartesian 3-vectors like $\boldsymbol{f}$ and $\boldsymbol{u}$ it doesn't matter whether the indices are up or down. We will denote their indices by symbols like $\boldsymbol{f}_i$ to distinguish against indices of $\mathbf{f}$, for instance.

Consider the first term, $f_0 = \gamma \boldsymbol{f} \cdot \boldsymbol{u}/c$. If we substitute in the 3-vector expression for the Lorentz force, we have $\gamma \boldsymbol{f} \cdot \boldsymbol{u}/c = \gamma q \boldsymbol{E} \cdot \boldsymbol{u}/c = \gamma q \boldsymbol{E}_i \boldsymbol{u}_i/c$. But the definition of $\mathbf{F}$ gives $f_0 = qF_{0\nu}u^\nu = q\gamma(F_{00}c + F_{01}u^1 + F_{02}u^2 + F_{03}u^3)$, so we have:

$$F_{0i} = \boldsymbol{E}_i/c$$

where we recall that $F_{00} = 0$ because of $\mathbf{F}$'s antisymmetry.

Now we find the rest of the components F_{ij}. We have $f_i = -\gamma \boldsymbol{f}_i$, and the full Lorentz force law is $\boldsymbol{f}_i = q(\boldsymbol{E}_i + \epsilon_{ijk}\boldsymbol{u}_j\boldsymbol{B}_k)$. By comparing with $f_i = qF_{i\nu}u^\nu$, we have:

$$-\gamma q(\boldsymbol{E}_i + \epsilon_{ijk}\boldsymbol{u}_j\boldsymbol{B}_k) = q\gamma(\underbrace{F_{i0}}_{-\boldsymbol{E}_i/c}c + F_{ij}\boldsymbol{u}^j)$$

$$F_{ij} = -\epsilon_{ijk}\boldsymbol{B}_k$$

$$[F_{\mu\nu}] = \begin{pmatrix} 0 & \boldsymbol{E}_1/c & \boldsymbol{E}_2/c & \boldsymbol{E}_3/c \\ -\boldsymbol{E_1}/c & 0 & -\boldsymbol{B}_3 & \boldsymbol{B}_2 \\ -\boldsymbol{E}_2/c & \boldsymbol{B_3} & 0 & -\boldsymbol{B}_1 \\ -\boldsymbol{E}_3/c & -\boldsymbol{B}_2 & \boldsymbol{B}_1 & 0 \end{pmatrix}$$

Contracting with $\eta^{\mu\nu}$, we find:

$$[F^\mu{}_\nu] = \begin{pmatrix} 0 & \boldsymbol{E}_1/c & \boldsymbol{E}_2/c & \boldsymbol{E}_3/c \\ \boldsymbol{E_1}/c & 0 & \boldsymbol{B}_3 & -\boldsymbol{B}_2 \\ \boldsymbol{E}_2/c & -\boldsymbol{B_3} & 0 & \boldsymbol{B}_1 \\ \boldsymbol{E}_3/c & \boldsymbol{B}_2 & -\boldsymbol{B}_1 & 0 \end{pmatrix} \qquad [F^{\mu\nu}] = \begin{pmatrix} 0 & -\boldsymbol{E}_1/c & -\boldsymbol{E}_2/c & -\boldsymbol{E}_3/c \\ \boldsymbol{E_1}/c & 0 & -\boldsymbol{B}_3 & \boldsymbol{B}_2 \\ \boldsymbol{E}_2/c & \boldsymbol{B_3} & 0 & -\boldsymbol{B}_1 \\ \boldsymbol{E}_3/c & -\boldsymbol{B}_2 & \boldsymbol{B}_1 & 0 \end{pmatrix}$$

From this final form, we can find out how electric and magnetic fields transform between frames, as $F'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma F^{\rho\sigma}$, so $[F'^{\mu\nu}] = \left[\Lambda^\mu{}_\rho\right][F^{\rho\sigma}][\Lambda^\nu{}_\sigma]$, which is

$$[F'^{\mu\nu}] = \begin{pmatrix} 0 & -\boldsymbol{E}_1/c & -\gamma(\boldsymbol{E}_2/c - \beta\boldsymbol{B}_3) & -\gamma(\boldsymbol{E}_3/c + \beta\boldsymbol{B}_2) \\ \boldsymbol{E}_1/c & 0 & -\gamma(\boldsymbol{B}_3 - \beta\boldsymbol{E}_2/c) & \gamma(\boldsymbol{B}_2 + \beta\boldsymbol{E}_3/c) \\ \gamma(\boldsymbol{E}_2/c - \beta\boldsymbol{B}_3) & \gamma(\boldsymbol{B}_3 - \beta\boldsymbol{E}_2/c) & 0 & -\boldsymbol{B}_1 \\ \gamma(\boldsymbol{E}_3/c + \beta\boldsymbol{B}_2) & -\gamma(\boldsymbol{B}_2 + \beta\boldsymbol{E}_3/c) & \boldsymbol{B}_1 & 0 \end{pmatrix}$$

which is simply the form of the Faraday tensor for different electric and magnetic fields:

$$\boldsymbol{E}' = \begin{pmatrix} \boldsymbol{E}_1 \\ \gamma(\boldsymbol{E}_2 - v\boldsymbol{B}_3) \\ \gamma(\boldsymbol{E}_3 + v\boldsymbol{B}_2) \end{pmatrix} \qquad \boldsymbol{B}' = \begin{pmatrix} \boldsymbol{B}_1 \\ \gamma(\boldsymbol{B}_2 + v\boldsymbol{E}_3/c^2) \\ \gamma(\boldsymbol{B}_3 - v\boldsymbol{E}_2/c^2) \end{pmatrix}$$

which gives the electric and magnetic fields as seen in different coordinate frames. As with the Lorentz transformations of spacetime coordinates, we see that the coordinates become mixed up with each other, however here we see that the x-directed components are unaffected, and it is the y- and z-components that are shuffled about; also there are now some + signs woven in.

5.3.2 4-Current $\mathbf{j}$

Consider a static charge density ρ as seen in $\mathcal{S}$. In a frame $\mathcal{S}'$ moving relative to $\mathcal{S}$ at a speed u, this charge density will appear to be greater by a factor of γ due to length contraction: $\rho' = \gamma\rho$. Also, in $\mathcal{S}'$, there will appear to be a current density $\boldsymbol{J}' = -\rho'\boldsymbol{u}$ (the minus sign is there as the charge will appear to be going in the opposite direction to the velocity of the frame itself, $\boldsymbol{u}$). In other words, we have transformed between the 4-vectors $(c\rho, \mathbf{0}) \to (\gamma c\rho, -\gamma\rho\boldsymbol{u})$. Justified by comparison with a Lorentz transformation of coordinates $(ct, \mathbf{0}) \to (\gamma ct, -\gamma\boldsymbol{u}t)$, we assert that $j^\mu = (c\rho, \boldsymbol{J})$ constitute the components of the *current 4-vector*.

The 3D continuity equation for charge is $\partial\rho/\partial t + \boldsymbol{\nabla}\cdot\boldsymbol{J} = 0$. Considering j^μ as a 4-vector, we see that $\partial_\mu j^\mu = \partial(\rho c)/\partial(ct) + \partial\boldsymbol{J}_i/\partial x^i = \partial\rho/\partial t + \boldsymbol{\nabla}\cdot\boldsymbol{J} = 0$. Thus the equation $\partial_\mu j^\mu = 0$ encodes conservation of charge.

5.3.3 Maxwell's Equations in Tensor Form

Consider the derivative $\partial_\nu F^{\nu 0}$. The first term is 0 as $F^{00} = 0$, so:

$$\partial_\nu F^{\nu 0} = \partial_i F^{i0} = \frac{1}{c}\boldsymbol{\nabla}\cdot\boldsymbol{E} = \frac{c\rho}{\epsilon_0 c^2} = \mu_0 c\rho = \mu_0 j^0$$

Inspired by this, we might suggest that $\partial_\nu F^{\nu\sigma} = \mu_0 j^\sigma$; let's check this for $\sigma = 1$:

$$\partial_\nu F^{\nu 1} = -\frac{1}{c}\frac{\partial\boldsymbol{E}_1}{\partial(ct)} + \frac{\partial\boldsymbol{B}_3}{\partial x^2} - \frac{\partial\boldsymbol{B}_2}{\partial x^3} = -\frac{1}{c^2}\frac{\partial\boldsymbol{E}_1}{\partial t} + (\boldsymbol{\nabla}\times\boldsymbol{B})_1 = \mu_0\boldsymbol{J}_1$$

where the last equality comes from Maxwell's 4th equation; one can easily check that the same results are found with $\sigma = 2, 3$. We have thus rewritten Maxwell's 1st and 4th equations as $\partial_\nu F^{\nu\sigma} = \mu_0 j^\sigma$. Furthermore, taking the divergence of this equation gives:

$$0 = \partial_\sigma\partial_\nu F^{\nu\sigma} = \mu_0\partial_\sigma j^\sigma$$

so charge conservation is built-in. The other two equations can be written as

$$\partial_{[\nu}F_{\sigma\rho]} = 0 \qquad \text{or} \qquad \partial_\nu F_{\sigma\rho} + \partial_\sigma F_{\rho\nu} + \partial_\rho F_{\nu\sigma} = 0$$

where the two are equivalent because $F_{\nu\sigma}$ is antisymmetric. This equation has lots of symmetry: choosing $(\nu, \sigma, \rho) = (0, 1, 2)$ is clearly the same as choosing $(1, 2, 0)$ or $(1, 0, 2)$. The independent choices are then $(0, 1, 2), (0, 1, 3), (0, 2, 3), (1, 2, 3)$. The latter is the only one which does not differentiate with respect to time, and so as we might expect it leads to Maxwell's 2nd law about the divergence of $\boldsymbol{B}$:

$$0 = \partial_{[1}F_{23]} \propto \partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = -\frac{\partial\boldsymbol{B}_1}{\partial x^1} - \frac{\partial\boldsymbol{B}_2}{\partial x^2} - \frac{\partial\boldsymbol{B}_3}{\partial x^3}$$

and so $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$. Choosing $(0, 1, 2)$ next, we find:

$$0 = \partial_{[0}F_{12]} \propto \partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01} = -\frac{\partial\boldsymbol{B}_3}{\partial(ct)} - \frac{1}{c}\frac{\partial\boldsymbol{E}_2}{\partial x^1} + \frac{1}{c}\frac{\partial\boldsymbol{E}_1}{\partial x^2}$$

which is $1/c$ times the 3rd component of $\boldsymbol{\nabla}\times\boldsymbol{E} = -\dot{\mathbf{B}}$; the remaining two possibilities simply give the other two components of this equation.

5.3.4 4-vector potential $\mathbf{A}$

If we take $F_{\nu\sigma} = \partial_\nu A_\sigma - \partial_\sigma A_\nu$, the cyclic property $\partial_{[\rho}F_{\nu\sigma]}$ is automatically satisfied by the symmetry of mixed partial derivatives, as is $\mathbf{F}$'s antisymmetry. However, the addition of a gradient of a scalar $\partial_\nu\psi$ to A_ν would not affect this $F_{\nu\sigma}$ if it is derived from A_ν in this way, so there is some some "gauge freedom" in defining A_ν. This is usually fixed by using the *Lorenz gauge* $\partial_\nu A^\nu = 0$, which is conveniently Lorentz invariant. We can then rewrite the "sourced" Maxwell equation $\partial_\nu F^{\nu\sigma} = \mu_0 j^\sigma$ in terms of the 4-vector potential. Lowering the indices of $F^{\nu\sigma}$,

$$\partial_\nu F^{\nu\sigma} = \partial_\nu \eta^{\nu\alpha}\eta^{\sigma\beta}F_{\alpha\beta} = \eta^{\nu\alpha}\eta^{\sigma\beta}(\partial_\nu\partial_\alpha A_\beta - \partial_\nu\partial_\beta A_\alpha) = \partial_\nu\partial^\nu A^\sigma - \partial_\beta \underbrace{\partial_\nu A^\nu}_{0}$$

if we use Lorenz gauge. We now write $\partial_\nu\partial^\nu = \Box$, where the $\Box$ symbol is the the *d'Alembertian*, or the Laplacian in Minkowski space, given by

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2$$

where $\boldsymbol{\nabla}^2$ is the 3D Laplacian. Anyway, this gives us $\partial_\nu F^{\nu\sigma} = \Box A^\sigma = \mu_0 j^\sigma$. In regions (or frames) where there is no current density, we have $\Box A^\sigma = 0$, which is a wave equation in A^σ for waves travelling at c.

The components of A^μ correspond to regular 3D potentials. For example, $F_{12} = \partial_1 A_2 - \partial_2 A_1$, so $-\boldsymbol{B}_3 = \partial_1 A_2 - \partial_2 A_1 = (\boldsymbol{\nabla}\times\boldsymbol{A})_3$. Similarly, $F_{01} = \partial_0 A_1 - \partial_1 A_0$, so $\boldsymbol{E}_1/c = \frac{1}{c}\,\partial A_1/\partial t - \partial A_0/\partial x$; for the rest of the components we similarly find $\boldsymbol{E} = -\boldsymbol{\nabla}\phi + \frac{1}{c}\,\partial\mathbf{A}/\partial t$. These are consistent with the identification:

$$A_\nu = (\phi/c, -\boldsymbol{A}) \qquad \Rightarrow \qquad A^\nu = (\phi/c, \boldsymbol{A})$$

where $\boldsymbol{A}$ is the regular 3-vector potential.

5.4 Beyond Minkowski Space

Minkowski space has a constant metric, so all the connection terms are all 0 everywhere, so $R_{abc}{}^d = R_{ab} = R = 0$: Minkowski space is flat. In this section we discuss some of the modifications that would need to be made to look at things in non-Minkowski spacetimes.

Consider a free particle. In an inertial frame, this free particle *by definition* has constant velocity $\boldsymbol{u}$, and hence constant γ. As such, if we parametrise by τ, the particle's (proper) time, we find:

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}u^\mu}{\mathrm{d}t} = 0$$

as the components $u^i = \gamma\boldsymbol{u}$ (as well as $u^0 = \gamma c$) do not change with t for a free particle in an inertial frame. The above is not a tensor equation; in curved spacetime it needs to be modified to:

$$\frac{Du^\mu}{D\tau} = 0$$

which *is* a tensor equation. But u^μ is the tangent vector to the curve that the free particle is travelling on, and so *it must travel along a geodesic* in curved spacetime.

For photons, $\mathbf{p} = \mathrm{d}\mathbf{x}/\mathrm{d}\zeta$ is a tangent vector to the path, and if the photon is "free", then we generalise to $Dp^\mu/D\zeta = 0$ and the photon also travels along a geodesic.

In a similar way to converting $\mathrm{d}u^\mu/\mathrm{d}\tau$ to $Du^\mu/D\tau$, similar modifications must be made to all equations used in this section involving d or ∂:

$$\mathrm{d}/\mathrm{d}\zeta \to D/D\zeta \qquad\qquad \partial_\mu \to \boldsymbol{\nabla}_\mu$$

for any parameters ζ or coordinates μ. For instance, the Maxwell tensor equations become:

$$\boldsymbol{\nabla}_\nu F^{\nu\sigma} = \mu_0 j^\sigma \qquad \boldsymbol{\nabla}_{[\nu}F_{\sigma\rho]} = 0$$

with $F_{\nu\sigma} = \boldsymbol{\nabla}_\nu A_\sigma - \boldsymbol{\nabla}_\sigma A_\nu$.

6 Gravitational Field Equations

6.1 Weak Field Limit

For weak, static gravitational fields, we expect the metric to look like

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

where each $|h_{\mu\nu}| \ll 1$, and $\partial_0 g_{\mu\nu} = \partial_0 h_{\mu\nu} = 0$. We also assume that any test masses are moving at much less than $c = \mathrm{d}ct/\mathrm{d}t = \mathrm{d}x^0/\mathrm{d}t$:

$$\left|\frac{\mathrm{d}x^i}{\mathrm{d}t}\right| \ll \frac{\mathrm{d}x^0}{\mathrm{d}t} \qquad \Rightarrow \qquad \left|\dot{x}^i\right| \ll \dot{x}^0$$

where the dots are with respect to τ, not t (we have multiplied through by $\mathrm{d}t/\mathrm{d}\tau$). The geodesic equation for a massive particle gives

$$0 = \ddot{x}^\mu + \Gamma^\mu_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma \approx \ddot{x}^\mu + \Gamma^\mu_{00}\dot{x}^0\dot{x}^0 = \ddot{x}^\mu + \Gamma^\mu_{00}\left(\frac{\mathrm{d}ct}{\mathrm{d}\tau}\right)^2$$

The connection terms are

$$\Gamma^\mu_{00} = \frac{1}{2}g^{\mu\nu}(\partial_0 g_{\nu 0} + \partial_0 g_{0\nu} - \partial_\nu g_{00}) = -\frac{1}{2}g^{\mu\nu}\partial_\nu h_{00} \approx -\frac{1}{2}\eta^{\mu\nu}\partial_\nu h_{00} = -\frac{1}{2}\eta^{\mu i}\partial_i h_{00}$$
$$= \begin{cases} 0 & \mu = 0 \\ \frac{1}{2}\partial^\mu h_{00} & \mu = i \end{cases}$$

Thus the geodesic equations become

$$0 = \ddot{x}^0 \qquad\qquad 0 = \ddot{x}^i + \frac{c^2}{2}\partial^i h_{00}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2$$

The first says that the quantity $\mathrm{d}t/\mathrm{d}\tau$ is constant; the second gives on multiplying through by this quantity's inverse

$$\frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2} = -\frac{c^2}{2}\boldsymbol{\nabla} h_{00}$$

We therefore identify $h_{00} = 2\Phi/c^2$, and thus $g_{00} = 1 + 2\Phi/c^2$.

6.2 Energy-Momentum Tensor $T^{\mu\nu}$

6.2.1 Dust

For an ensemble of particles not moving relative to each other ("dust"), the energy density is $\rho_0 c^2 = n_0 m c^2$ in the rest frame, and $(\gamma n_0)(\gamma m c^2) = \gamma^2\rho_0 c^2$ in a relatively moving frame. If we define $T^{\mu\nu} = \rho_0 u^\mu u^\nu$, where $\mathbf{u}$ is the 4-velocity of the dust particles, then T^{00} is the energy density even after a Lorentz transformation, where $u^0 = c$ becomes $u^0 = \gamma c$. As ρ_0, the density in the rest frame, is a scalar field, $T^{\mu\nu}$ is a symmetric tensor field.

Consider the frame in which the dust is moving at $\boldsymbol{u}$, so $u^\mu = \gamma(c, \boldsymbol{u})$. In this frame, the components T^{0i} are given by

$$T^{0i} = \gamma^2\rho_0 c\boldsymbol{u}^i = c\,\underbrace{(\gamma n_0)(\gamma m\boldsymbol{u}^i)}_{\text{momentum density}} = \underbrace{(\gamma n_0)(\gamma m c^2}_{\text{energy density}}\,\boldsymbol{u}^i)/c$$

T^{0i} is thus as a momentum density/an energy density flux. Finally, T^{ij} is given by

$$T^{ij} = \gamma^2 \rho_0 \boldsymbol{u}^i \boldsymbol{u}^j = \underbrace{(\gamma n_0)(\gamma m \boldsymbol{u}^i)}_{i\text{-momentum density}} \boldsymbol{u}^j$$

so T^{ij} is identified as i-momentum flux in the j-direction, or vice versa.

6.2.2 Ideal Fluids

For ideal fluids, $\exists$ a frame for which $T^{0i} = 0$ (no energy flow/conduction) and $T^{ij} \propto \delta^{ij}$ (isotropic pressure). In such a frame, $T^{ij} = \mathfrak{diag}(\rho_0 c^2, p_0, p_0, p_0)$, where p_0 is the pressure in the rest frame, a scalar field. More generally,

$$T^{\mu\nu} = \left(\rho_0 + \frac{p_0}{c^2}\right) u^\mu u^\nu - p_0 g^{\mu\nu}$$

which reduces to the diagonal form in the rest frame and the limit of Minkowski space. It is common for $\rho_0 c^2 \gg p_0$.

6.2.3 Conservation of Energy-Momentum

Conservation of energy and momentum is written $\boldsymbol{\nabla}_\mu T^{\mu\nu} = 0$. In local inertial coordinates,

$$\frac{\partial}{\partial ct} \underbrace{[\text{energy density}]}_{T^{00}} + \boldsymbol{\nabla} \cdot \underbrace{[\text{energy density} \times \boldsymbol{u}]/c}_{T^{0i}} = 0 \qquad \text{(cons. } E/c\text{)}$$

$$\frac{\partial}{\partial ct} \underbrace{(c[i\text{-momentum density}])}_{T^{0i}} + \boldsymbol{\nabla} \cdot \underbrace{[i\text{-momentum density} \times \mathbf{u}]}_{T^{ij}} = 0 \qquad \text{(cons. } \boldsymbol{p}^{\boldsymbol{i}}\text{)}$$

Applying $\boldsymbol{\nabla}_\mu T^{\mu\nu}$ to the energy-momentum tensor of an ideal fluid recovers the continuity and Euler equations of fluid mechanics.

6.3 Einstein Field Equations

If small masses/pressures are involved, we have approximately

$$g_{00} = 1 + \frac{2\Phi}{c^2} \qquad T_{00} = \rho_0 c^2 \qquad \boldsymbol{\nabla}^2 \Phi = 4\pi G \rho_0$$

$$\Rightarrow \boldsymbol{\nabla}^2 g_{00} = \frac{8\pi G}{c^2} \rho_0 = \frac{8\pi G}{c^4} T_{00}$$

$\boldsymbol{\nabla}^2 g_{00}$ is (vaguely speaking) a measure of the manifold's curvature, so we expect an equation of the form $K_{\mu\nu} = T_{\mu\nu}$ where $K_{\mu\nu}$ has something to do with the curvature, is symmetric, and divergenceless (as $\boldsymbol{\nabla}^\mu T_{\mu\nu} = 0$). The Einstein tensor, plus an arbitrary multiple of the metric, satisfies all these requirements, so we expect

$$G_{\mu\nu} + \Lambda g_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu}$$

for some κ which we derive below. Λ has been measured to be around $(3.04\text{Gpc})^{-2}$, so will only be relevant on large scales.

For negligible pressures, $T_{00} = \rho_0 c^2$ and $T \equiv g^{\mu\nu} T_{\mu\nu} = \rho_0 c^2$. Neglecting Λ, and taking the trace of the above equation, using $g^{\mu\nu} g_{\mu\nu} = \delta^\mu_\mu = 4$, we find $-R = -\kappa \rho_0 c^2$. Substituting back and focusing on the 00-component,

$$R_{00} - \frac{\kappa}{2}\rho_0 c^2 = -\kappa \rho_0 c^2 \qquad \Rightarrow \qquad R_{00} = -\frac{\kappa}{2}\rho_0 c^2$$

This component of the curvature is given in the stationary weak-field limit by

$$R_{00} \equiv R_{c00}{}^{c} = --\partial_c \Gamma^c_{00} + \partial_0 \Gamma^c_{c0} + \Gamma^e_{c0}\Gamma^c_{0e} - \Gamma^e_{00}\Gamma^c_{ce} \approx -\partial_c \Gamma^c_{00} = -\partial_i \Gamma^i_{00} = -\frac{1}{2}\nabla^2 h_{00} = -\frac{1}{c^2}\nabla^2 \Phi$$

Thus

$$\nabla^2 \Phi = \frac{\kappa}{2} c^4 \rho_0 = 4\pi G \rho_0 \qquad \Rightarrow \qquad \kappa = \frac{8\pi G}{c^4}$$

So the EFE is

$$G_{\mu\nu} + \Lambda g_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}$$

There is an equivalent formulation with $R_{\mu\nu}$ in terms of $T^{\mu\nu}$ and T rather than the other way around. Tracing,

$$-R + 4\Lambda = -\frac{8\pi G}{c^4} T \qquad \Rightarrow \qquad \begin{aligned} R_{\mu\nu} &= -\frac{8\pi G}{c^4} T_{\mu\nu} + \frac{1}{2}\left(\frac{8\pi G}{c^4} T + 4\Lambda\right) g_{\mu\nu} - \Lambda g_{\mu\nu} \\ &= -\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right) + \Lambda g_{\mu\nu} \end{aligned}$$

7 The Schwarzschild Solution

The Schwarzschild solution is a vacuum solution of EFE, that is, of $G_{\mu\nu} = 0$ and thus $R_{\mu\nu} = 0$. It is the most general vacuum solution which is static and spherically symmetric.

$$\boxed{\mathrm{d}s^2 = c^2\left(1 - \frac{2\mu}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2\mu}{r}\right)^{-1}\mathrm{d}r^2 - r^2\,\mathrm{d}\Omega^2}$$

$$\mu = \frac{GM}{c^2} \qquad\qquad \mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$$

- This is only the metric *outside* the mass – within the mass, $G_{\mu\nu} \neq 0$ so the Schwarzschild solution is invalid and a different solution is needed instead
- It is complicated to define distances from the origin, so the r coordinate has a careful interpretation: surfaces of constant t and r are spheres of area $4\pi r^2$
- As $r \to \infty$, the coordinates regain their usual interpretations and the metric tends to $\eta_{\mu\nu}$
- There appear to be two singularities to the metric:
  - $r = r_s \equiv 2\mu$: This is not a true singularity, only an illusory *coordinate singularity* – it can be removed by a different choice of coordinates.
  - $r = 0$: This is a genuine singularity and the spacetime sorta breaks here.

  Singularity of a point can be deduced by considering a coordinate-independent quantity, such as $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$. This is proportional to r^{-6}, so $r = 2\mu$ is actually fine.

7.1 Schwarzschild Orbits

The paths taken by particles around spherically symmetric masses are geodesics of the Schwarzschild metric, and thus extremes of an action with Lagrangian

$$\mathcal{L} = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = c^2\left(1-\frac{2\mu}{r}\right)\dot{t}^2 - \left(1-\frac{2\mu}{r}\right)^{-1}\dot{r}^2 - r^2\left(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2\right)$$

The overdots are with respect to ζ, which represents either proper time τ if we are dealing with a massive particle, or some other parameter if with a massless particle. As two of the coordinates θ and ϕ have different dimensions to r, we no longer bother to use ct as a coordinate and just write $x^0 = t$.

In addition, this Lagrangian is independent of the parameter and so is a constant. For a massive particle, parametrised by τ, $\mathcal{L} = |\mathbf{u}|^2 = c^2$. For a massless particle, their motion in the manifold is along the light cone and they travel along null paths, so $\mathcal{L} = |\mathbf{p}|^2 = 0$.

$e - \mathcal{L}\{\theta\}$:

$$2r^2\sin\theta\cos\theta\dot{\phi}^2 = 2\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(r^2\dot{\theta}\right)$$

which is solved by $\theta = \pi/2$.

$e - \mathcal{L}\{\phi\}$:

$$0 = \frac{\mathrm{d}}{\mathrm{d}\zeta}\left(2r^2\sin^2\theta\dot{\phi}\right)$$

with $\theta = \pi/2$, this says that $r^2\dot{\phi}$ is a constant, h, familiar as angular momentum per mass.

$e - \mathcal{L}\{t\}$:

$$0 = \frac{\mathrm{d}}{\mathrm{d}\zeta}\left[c^2\left(1-\frac{2\mu}{r}\right)\dot{t}\right]$$

so $(1-2\mu/r)\dot{t}$ is a constant, k, which turns out to be related to the particle's energy.

Rather than using the $e-\mathcal{L}\{r\}$, it turns out to be easier to use the first integral $\mathcal{L} = \{c^2, 0\}$:

$$\begin{aligned}
\mathcal{L} &= c^2\left(1-\frac{2\mu}{r}\right)\dot{t}^2 - \left(1-\frac{2\mu}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 \\
&= c^2\left(1-\frac{2\mu}{r}\right)^{-1}k^2 - \left(1-\frac{2\mu}{r}\right)^{-1}\dot{r}^2 - \frac{h^2}{r^2} \\
\left(1-\frac{2\mu}{r}\right)\mathcal{L} &= c^2k^2 - \dot{r}^2 - \frac{h^2}{r^2}\left(1-\frac{2\mu}{r}\right) \\
\frac{1}{2}(c^2k^2-\mathcal{L}) &= \frac{1}{2}\dot{r}^2 - \frac{\mu}{r}\mathcal{L} + \frac{h^2}{2r^2}\left(1-\frac{2\mu}{r}\right)
\end{aligned}$$

We thus have a sort of energy equation (though recall that r and $\dot{r}$ only *correspond* to radial distance and its time derivative).

7.1.1 Massive Particles: $\mathcal{L} = c^2$

$$\frac{1}{2}\dot{r}^2 - \frac{GM}{r} + \frac{h^2}{2r^2}\left(1-\frac{2\mu}{r}\right) = \frac{1}{2}c^2(k^2-1) \qquad V_{\text{eff}}(r) = -\frac{GM}{r} + \frac{h^2}{2r^2}\left(1-\frac{2\mu}{r}\right)$$

like the Newtonian effective potential, but with a collapsing centrifugal barrier. The form of the effective potential depends on the ratio $\bar{h} \equiv h/\mu c$ as shown in Figure 3.

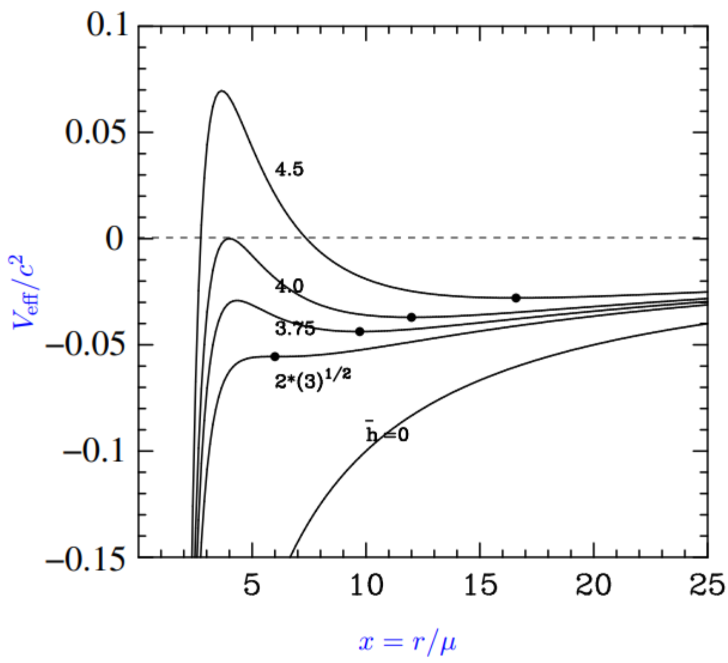

Figure 3 | Form of V_{eff} for different values of $\bar{h}$.

Setting $\mathrm{d}V_{\text{eff}}/\mathrm{d}r = 0$ gives two roots, at

$$\frac{r}{\mu} = \frac{\bar{h}^2}{2}\left(1 \pm \sqrt{1 - \frac{12}{\bar{h}^2}}\right)$$

from which we see that

- $\bar{h} < \sqrt{12}$: no orbits exist, V_{eff} is monotonically positive, and everything falls in.
- $\bar{h} = \sqrt{12}$: one semi-stable orbit at $r = 6\mu$, the *innermost stable circular orbit* or *ISCO*.
- $\bar{h} > \sqrt{12}$: two turning points, one on either side of $r = 6\mu$. Using $\mathrm{d}^2V_{\text{eff}}/\mathrm{d}r^2$, we find that the inner root is unstable and the outer root stable.
- $\bar{h} \to \infty$: stable root tends to ∞; unstable root tends to $r = 3\mu$.

7.1.2 Massless Particles: $\mathcal{L} = 0$

$$\frac{1}{2}\dot{r}^2 + \frac{h^2}{2r^2}\left(1 - \frac{2\mu}{r}\right) = \frac{1}{2}c^2k^2 \qquad\qquad V_{\text{eff}}(r) = \frac{h^2}{2r^2}\left(1 - \frac{2\mu}{r}\right)$$

The form of V_{eff} does not depend on h: there is a single unstable orbit at $r = 3\mu$, where $V_{\text{eff}} = h^2/54\mu^2$. If $c^2k^2/2$ is less than this, then the particle can never reach the maximum ($\dot{r}^2$ would be negative here).

The impact parameter of an orbit can be derived by converting the energy equation into an orbit-shape equation by writing

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 = \left(\frac{\dot{r}}{\dot{\phi}}\right)^2 = \frac{r^4}{h^2}\dot{r}^2 = \frac{r^4}{h^2}\left[c^2k^2 - \frac{h^2}{r^2}\left(1 - \frac{2\mu}{r}\right)\right] = r^2\left[\frac{c^2k^2}{h^2}r^2 - 1 + \frac{2\mu}{r}\right]$$

where we have substituted $\dot{r}^2$ from the energy equation. As $r \to \infty$, we expect the path to be a straight line, with impact parameter b. This line is described by

$$r = \frac{b}{\sin\phi} \qquad \Rightarrow \qquad \left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 = \left(-\frac{b}{\sin^2\phi}\cos\phi\right)^2 = \frac{r^4}{b^2}\left(1 - \frac{b^2}{r^2}\right) = r^2\left(\frac{r^2}{b^2} - 1\right)$$

and so from the large-r limit of the orbit equation above we can identify $b = h/ck$. As such, if light passes with an impact parameter of less than $\sqrt{27}\mu$, it will be captured.

7.1.3 Energy

Working first more generally than the Schwarzschild metric, thinking of any static (not necessarily spherically symmetric) metric with

$$\mathrm{d}s^2 = g_{00}\,\mathrm{d}t^2 + g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j \qquad \Rightarrow \qquad \mathcal{L} = g_{00}\dot{t}^2 + g_{ij}\dot{x}^i\dot{x}^j$$

Taking $e - \mathcal{L}\{t\}$ gives:

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(2g_{00}\dot{t}\right) = 0 \qquad \Rightarrow \qquad g_{00}\dot{t} = kc^2$$

where we have chosen the arbitrary constant to be consistent with what we had above for the Schwarzschild metric.

The four-velocity has square-magnitude $g_{\mu\nu}u^\mu u^\nu = c^2$ as before. Consider a stationary observer. It will observe its *own* three-velocity to be $\mathbf{0}$, that is $u^i = 0$. As for u^0, we have

$$c^2 = g_{\mu\nu}u^\mu u^\nu = g_{00}\left(u^0\right)^2 \qquad \Rightarrow \qquad u^\mu = \left(\frac{c}{\sqrt{g_{00}}}, \mathbf{0}\right)$$

Now the energy E of a particle (as observed by some observer at the same point) which has momentum $\mathbf{p}$ is given by $E = g_{\mu\nu}u^\mu p^\nu$, a quite confusing expression relating the energy of the *particle*, the four-velocity of the *observer* and the four-momentum of the *particle*. This is because, in local inertial coordinates ($g_{\mu\nu} = \eta_{\mu\nu}$) in which the observer is at rest ($u^\mu = (c, \mathbf{0})$), the component $p^0 = E/c$, so $g_{\mu\nu}u^\mu p^\nu = \eta_{00}c(E/c) = E$. Back in general (non-inertial) coordinates in which the observer is at rest, we then find

$$E = g_{\mu\nu}u^\mu p^\nu = g_{0\nu}u^0 p^\nu = g_{00}u^0 p^0 = \sqrt{g_{00}}cp^0$$

from the point of view of this observer. For a massive particle, $p^0 = m\dot{x}^0 = m\dot{t}$ where differentiation is with respect to τ; for a massless particle $p^0 = \dot{x}^0 = \dot{t}$ with respect to some ζ. For the massive case,

$$E = \sqrt{g_{00}}mc\dot{t} = \frac{kmc^3}{\sqrt{g_{00}}}$$

and the massless case is the same expression but with $m = 1$ (not to say that massless particles have $m = 1$; the differentiation was with respect to a different variable $\zeta \neq \tau$). This is the sum of all the particle's energy: rest-mass, kinetic, and potential. For the Schwarzschild metric,

$$E = \begin{cases} kmc^2(1 - 2\mu/r)^{-1/2} & \text{massive} \\ kc^2(1 - 2\mu/r)^{-1/2} & \text{massless} \end{cases}$$

7.1.4 Gravitational Redshift

For a photon, $E = h\nu$, where E and ν are the values observed by the same observer at a given point. Consider a photon emitted from a certain point A with frequency ν_A as observed by an observer at A, received at B with ν_B as observed by an observer at B, *where both observers are stationary* relative to the mass. The ratio of the two frequencies is given by

$$\frac{\nu_B}{\nu_A} = \frac{E_B}{E_A} = \sqrt{\frac{g_{00}(A)}{g_{00}(B)}}$$

which for the Schwarzschild case gives

$$\frac{\nu_B}{\nu_A} = \sqrt{\frac{1 - 2\mu/r_A}{1 - 2\mu/r_B}}$$

If we are observing at $r_B = \infty$ and recall that $1 + z = \lambda/\lambda_0 = \nu_A/\nu_B$, we have

$$1 + z = \frac{1}{\sqrt{1 - 2\mu/r_A}}$$

Which $\to \infty$ as $r_A \to 2\mu\ldots$

7.2 Schwarzschild Black Holes

For $r < r_s = 2\mu$, $g_{00} < 0$, and so any vector pointing along the time direction (const, r, θ, ϕ) becomes spacelike; conversely spatial basis vectors become timelike. A particle's worldline has to be timelike (otherwise it travels faster than the speed of light), so a particle must move in space[4] and cannot stay still.

Henceforth we restrict ourselves to radial motion, setting $\mathrm{d}\Omega = 0$, effectively taking a slice through the manifold in the $t - r$ plane. Light cones are bounded by null geodesics, so to find out how causality works here we need to find these. We have

$$0 = \mathrm{d}s^2 = c^2\left(1 - \frac{2\mu}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2\mu}{r}\right)^{-1}\mathrm{d}r^2 \qquad \Rightarrow \qquad \frac{\mathrm{d}ct}{\mathrm{d}r} = \pm\left(1 - \frac{2\mu}{r}\right)^{-1}$$

the solutions of which are

$$ct = \pm\left(r + 2\mu\ln\left|\frac{r}{2\mu} - 1\right|\right) + \text{const.}$$

related to each other by time reversal $t \to -t$. Note that as $t \to \infty$, $r \to \pm ct$ as in Minkowski (the linear term dominates the logarithmic one), corresponding to the paths of ingoing ($-$) and outgoing ($+$) massless particles. The null geodesics are plotted in Figure 4 below.

t refers to the time that an observer at ∞ experiences. We see that infalling ($-$) particles will reach $r = 2\mu$ only at infinite t, so on watching something fall in, it will fall apparently slower and slower (appearing redder and redder). However, from the particle's point of view,

[4] It *may* also move in time, though this is not essential. This is like how in normal regions of spacetime particles must move in time, and *may* also move in space (so long as they don't move too fast)

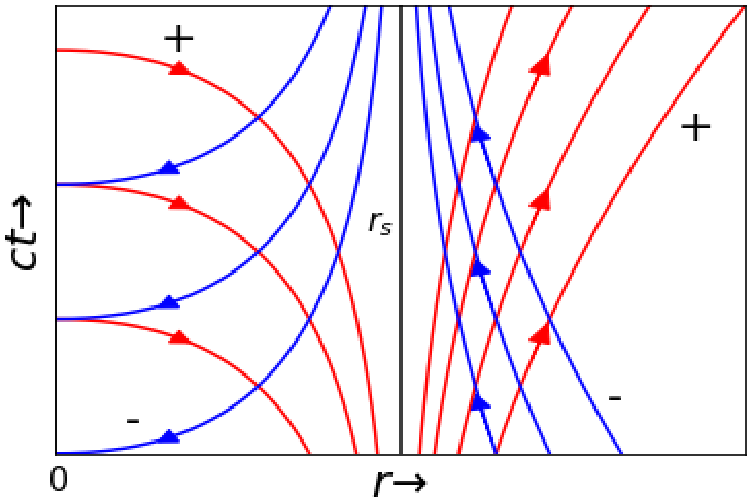

Figure 4 | Null radial geodesics in the Schwarzschild metric. Null radial geodesics bound particles' lightcones. The null geodesics with the + sign are plotted in red; those with − are in blue. Arrows indicate the direction of increasing ζ along the path. They both tend to $ct = \pm r$ at large r.

it will pass through the surface $r = r_s$ in a finite amount of time (from this perpective, τ). Using the orbit equation with $h = 0$ (radial), and for the simple case $k = 1$,

$$\frac{1}{2}\dot{r}^2 = \frac{GM}{r} \qquad \Rightarrow \qquad \dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\tau} = -\sqrt{\frac{2GM}{r}} = -\sqrt{\frac{2\mu c^2}{r}}$$

Setting $r(\tau_0) = r_0$, we integrate to obtain

$$\tau - \tau_0 = -\int_{r_0}^{r} \sqrt{\frac{r}{2\mu c^2}}\,\mathrm{d}r = -\frac{2}{3\sqrt{2\mu c^2}}\left(r^{3/2} - r_0^{3/2}\right)$$

so if we set $r = r_s = 2\mu$, the τ taken between being at r_s and r_0 is finite. However, the t taken (as observed at infinity) is infinite:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\dot{r}}{\dot{t}} = -\sqrt{\frac{2\mu c^2}{r}}\left(1 - \frac{2\mu}{r}\right)$$

Integrating to

$$t - t_0 = \int_{r_0}^{r} \sqrt{\frac{r}{2\mu c^2}} \frac{\mathrm{d}r}{1 - \frac{2\mu}{r}} = -\frac{2\mu}{c}\int_{r_0/2\mu}^{r/2\mu} \frac{x^{3/2}}{x-1}\,\mathrm{d}x = -\frac{2\mu}{c}\left[\frac{2}{3}\left(x^{3/2} + 3x^{1/2}\right) + \ln\left|\frac{x^{1/2}-1}{x^{1/2}+1}\right|\right]_{r_0/2\mu}^{r/2\mu}$$

where the last step can be done by substituting $x = u^2$ and doing partial fractions at it. We see that letting $r \to 2\mu$ from above gives a divergence, so the body takes an infinite amount of t to reach the singularity.

7.2.1 Within the Schwarzschild Radius

The derivative of time with respect to the geodesic parameter (whether τ or ζ) is $\dot{t} = k(1 - 2\mu/r)^{-1}$, so for $r < 2\mu$, $\dot{t} < 0$. Infalling (−) observers, which have $\mathrm{d}r/\mathrm{d}t > 0$ as shown in Figure

4, thus have $\mathrm{d}r/\mathrm{d}\tau < 0$. Hence the path travelled by them is in the direction of negative r and negative t: if an external observer and an internal observer could communicate (which they can't), *they would see each other moving backwards in time.*

Recall that A can only affect B if they are connected by a timelike or null curve; otherwise A would be doing something that moves faster than light. Whereas timelike curves are normally more directed along the time axis than the spatial axes, here the reverse is true: the lightcones are more directed towards and away from the singularity, than along the time axis. For particle that has fallen through $r = r_s$, it can only affect things that have a smaller r than it, even travelling at the speed of light. Conversely the only things that can affect it are at a *larger* r than it: the *past* lightcone within $r < r_s$ is pointed outwards, in the direction of increasing r.

7.2.2 Eddington-Finkelstein Coordinates

Much of the difficulty here comes from t. Infalling photons follow ingoing geodesics

$$ct = -\left(r + 2\mu \ln\left|\frac{r}{2\mu} - 1\right|\right) + \text{const.}$$

so we define a new coordinate t' by

$$ct' \equiv ct + 2\mu \ln\left|\frac{r}{2\mu} - 1\right|$$

and so infalling photons follow the paths $ct' = -r+\text{const}$. Now $c\,\mathrm{d}t' = c\,\mathrm{d}t + (r/2\mu - 1)^{-1}\,\mathrm{d}r$, so substituting into the Schwarzschild line element we have

$$\begin{aligned}\mathrm{d}s^2 &= c^2\left(1 - \frac{2\mu}{r}\right)\left[\mathrm{d}t' - \frac{1}{c}\left(\frac{r}{2\mu} - 1\right)^{-1}\mathrm{d}r\right]^2 - \left(1 - \frac{2\mu}{r}\right)^{-1}\mathrm{d}r^2 - r^2\,\mathrm{d}\Omega^2 \\ &= c^2\left(1 - \frac{2\mu}{r}\right)\mathrm{d}t'^2 - \frac{4\mu c}{r}\,\mathrm{d}t'\,\mathrm{d}r - \left(1 + \frac{2\mu}{r}\right)\mathrm{d}r^2 - r^2\,\mathrm{d}\Omega^2\end{aligned}$$

from which we see that there is no true singularity at $r = 2\mu$, thought the true singularity at $r = 0$ is still apparent. This metric is also non-diagonal.

One can also construct *outgoing* Eddington-Finkelstein coordinates:

$$ct^* = ct - 2\mu \ln\left|\frac{r}{2\mu} - 1\right|$$

so that outgoing photons travel along $ct^* = r+\text{const}$. The line element then becomes

$$\mathrm{d}s^2 = c^2\left(1 - \frac{2\mu}{r}\right)\mathrm{d}t^{*2} + \frac{4\mu c}{r}\,\mathrm{d}t^*\,\mathrm{d}r - \left(1 + \frac{2\mu}{r}\right)\mathrm{d}r^2 - r^2\,\mathrm{d}\Omega^2$$

7.2.3 Gravitational Collapse

Consider a cloud of dust contracting to form a black hole. Assume the dust has no pressure and so there are no forces at play other than gravity. A particle on the edge of the cloud experiences a Schwarzschild spacetime, as the rest of the star appears spherically symmetric. The star may have $r > 2\mu$ to start with, but the spacetime does not change according to Birkhoff's theorem.

Suppose the particle emits a light ray at coordinates (t_E, r_E) which is received by a stationary observer (t_R, r_R) where r_R is constant. We wish to know t_R as a function of r_E. Both of these points are on the outgoing null geodesic traversed by the light ray, so

$$ct_R \underbrace{-r_R - 2\mu \ln\left|\frac{r_R}{2\mu} - 1\right|}_{\text{const.}} = ct_E - r_E - 2\mu \ln\left|\frac{r_E}{2\mu} - 1\right|$$

Now ct_E and r_E are related by that complicated expression from earlier with the $\ln \frac{\sqrt{x}-1}{\sqrt{x}+1}$ in it. As $r \to 2\mu$, only this ln term will be dominant, so we have

$$ct_E = 2\mu \ln\left(\frac{\sqrt{r_E/2\mu}+1}{\sqrt{r_E/2\mu}-1}\right)$$

which on substitution into the above gives

$$\begin{aligned}
ct_R + \text{const.} &\approx 2\mu \ln\left(\frac{\sqrt{r_E/2\mu}+1}{\sqrt{r_E/2\mu}-1}\right) - 2\mu \ln\left(\frac{r_E}{2\mu} - 1\right) \\
&= 2\mu \ln\left(\frac{\sqrt{r_E/2\mu}+1}{\sqrt{r_E/2\mu}-1}\frac{1}{\left(\sqrt{r_E/2\mu}+1\right)\left(\sqrt{r_E/2\mu}-1\right)}\right) \\
&= -4\mu \ln\left(\sqrt{\frac{r_E}{2\mu}} - 1\right) \approx -4\mu \ln\left(\frac{r_E}{2\mu} - 1\right)
\end{aligned}$$

where the final approximation comes from I have no idea where. Anyway this can be rearranged to

$$r_E = 2\mu + \text{const.} \times e^{-ct_R/4\mu}$$

giving an exponentially decaying distance between the particle and the Schwarzschild radius, with time constant $4\mu/c$.

We can also deduce the redshift of the light emitted, as observed by a stationary receiver. You might think that we can just use the formula from earlier, but that assumed that the emitter and receiver were stationary with respect to the mass; here the emitter is moving towards the black hole and the receiver is stationary, so there will be Doppler redshift as well as gravitational redshift. As before, $E = g_{\mu\nu}u^\mu p^\nu$, and for the emitter, relative to the mass. Supposing again that $k = 1$, we have

$$u_E^0 = \dot{t} = \left(1 - \frac{2\mu}{r}\right)^{-1} \qquad\qquad u_E^1 = \dot{r} \equiv \frac{\mathrm{d}r}{\mathrm{d}\tau} = -\sqrt{\frac{2\mu c^2}{r}}$$

and $u_E^2 = \dot{\theta} = u_E^3 = \dot{\phi} = 0$; this four-velocity is also normalised to $g_{\mu\nu}u^\mu u^\nu = c^2$.

We now require the four-momentum of the photon as a function of r. This will be $p^\mu = \dot{x}^\mu = (\dot{t}, \dot{r}, 0, 0)$ where differentiation is now with respect to some general affine parameter ζ. We have $\dot{t} = k(1 - 2\mu/r)^{-1}$ (where this $k \neq 1$ is that of the photon, not the dust particle). From the orbit equation for a massless particle with $h = 0$, we find simply $\dot{r} = ck$. Thus

$$p^\mu = k\left(\left(1 - \frac{2\mu}{r}\right)^{-1}, c, 0, 0\right)$$

which can be shown to be normalised $g_{\mu\nu}p^\mu p^\nu = 0$. The energy of the emitted photon is then given by

$$
\begin{aligned}
E_E = g_{\mu\nu}u_E^\mu p^\nu &= c^2\left(1-\frac{2\mu}{r}\right)\left(1-\frac{2\mu}{r}\right)^{-1} k\left(1-\frac{2\mu}{r}\right)^{-1} - \left(1-\frac{2\mu}{r}\right)^{-1}\left[-\sqrt{\frac{2\mu c^2}{r}}\right] kc \\
&= kc^2\left(1-\frac{2\mu}{r}\right)^{-1}\left[1+\sqrt{\frac{2\mu}{r}}\right] = kc^2\left(1-\sqrt{\frac{2\mu}{r}}\right)^{-1}
\end{aligned}
$$

evaluated at r_E. The energy of the received photon is then $E_R = g_{\mu\nu}u_R^\mu p^\nu$. Being stationary[5],

$$
u_R^\mu = \left(\left(1-\frac{2\mu}{r}\right)^{-1/2}, 0, 0, 0\right)
$$

which has simply been deduced by normalisation. We then have:

$$
E_R = g_{\mu\nu}u_R^\mu p^\nu = c^2\left(1-\frac{2\mu}{r}\right)\left(1-\frac{2\mu}{r}\right)^{-1/2} k\left(1-\frac{2\mu}{r}\right)^{-1} = kc^2 = kc^2\left(1-\frac{2\mu}{r}\right)^{-1/2}
$$

evaluated at r_R.

The frequencies are simply the energies divided by Planck's constant, so the ratio of frequencies is the ratio of energies. Thus

$$
1+z = \frac{\nu_E}{\nu_R} = \frac{E_E}{E_R} = \frac{\left(1-\sqrt{2\mu/r_E}\right)^{-1}}{(1-2\mu/r_R)^{-1/2}} \to \left(1-\sqrt{\frac{2\mu}{r_E}}\right)^{-1}
$$

as $r_R \to \infty$.

7.3 Experimental Tests of General Relativity

7.3.1 Precession

Recall that for a massive particle

$$
\frac{1}{2}\dot{r}^2 - \frac{GM}{r} + \frac{h^2}{2r^2}\left(1-\frac{2\mu}{r}\right) = \frac{1}{2}c^2(k^2-1)
$$

Taking interest in the orbit shape, we substitute

$$
\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\phi}\dot{\phi} = \frac{h}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\phi} = -h\frac{\mathrm{d}u}{\mathrm{d}\phi} \qquad \text{(u=1/r)}
$$

Substituting this, changing coordinates to u and dividing through by h^2 gives

$$
\frac{1}{2}\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 - \frac{GM}{h^2}u + \frac{1}{2}u^2(1-2\mu u) = \frac{c^2(k^2-1)}{2h^2}
$$

[5] This four-velocity appears to contrast with the above where $\dot{t} = k(1-2\mu/r)^{-1}$, but that was in fact derived from the Lagrangian procedure, so is only true along geodesics. The path of the receiver is **not** a geodesic for finite r_R, as the receiver is not in freefall – otherwise it would be falling! It must have rocket boosters or something to prevent it from falling towards the mass.

Differentiating and dividing through by $\mathrm{d}u/\mathrm{d}\phi$:

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} - \frac{GM}{h^2} + u - 3\mu u^2 = 0$$

which is the same as the Newtonian equation except for the $3\mu u^2$ term. In terms of the dimensionless $U \equiv uh^2/GM = r_0 u$, we multiply through by r_0:

$$\frac{\mathrm{d}^2U}{\mathrm{d}\phi^2} - 1 + U = 3\frac{GM}{c^2}\frac{U^2}{r_0} = \underbrace{3\left(\frac{GM}{hc}\right)^2}_{\alpha} U^2$$

This entire equation is dimensionless and α is typically small – for the orbit of Mercury $\alpha \sim 10^{-7}$. Thus seek an solution $U(\phi) = U_0(\phi) + \alpha U_1(\phi)$ accurate to first order in α, where $U_0(\phi) = 1 + e\cos\phi$ is the solution with $\alpha = 0$. Substituting this form of $U(\phi)$:

$$\begin{aligned}
\alpha\frac{\mathrm{d}^2U_1}{\mathrm{d}\phi^2} + \alpha U_1 = \alpha(1+e\cos\phi)^2 \qquad &\Rightarrow \qquad U_1'' + U_1 = 1 + \frac{e^2}{2} + 2e\cos\phi + \frac{e^2}{2}\cos 2\phi \\
&\Rightarrow \qquad U_1(\phi) = 1 + \frac{e^2}{2} + e\phi\sin\phi - \frac{e^2}{6}\cos 2\phi \\
&\qquad\qquad\qquad \approx e\phi\sin\phi
\end{aligned}$$

where we keep only the non-periodic term as the others will be eternally small whereas this will eventually become decently large. Thus

$$\begin{aligned}
U(\phi) &\approx 1 + e\cos\phi + e\alpha\phi\sin\phi \approx 1 + e\big[\cos(\alpha\phi)\cos\phi + \sin(\alpha\phi)\sin\phi - \mathcal{O}(\alpha\phi)^2\big] \\
&= 1 + e\cos\left((1-\alpha)\phi\right)
\end{aligned}$$

ϕ thus needs to increase by more than 2π (in fact $2\pi/(1-\alpha)$) for the cosand to be 0 again and the orbit to reach another perihelion. This is more than expected by an amount

$$\frac{2\pi}{1-\alpha} - 2\pi \approx 2\pi\alpha = 6\pi\left(\frac{GM}{hc}\right)^2$$

For Mercury, this matches the discrepancy observed between Newtonian gravity (including Newtonian interactions with other planets) and what is observed.

7.3.2 Gravitational Lensing

For a massless particle we have instead

$$\begin{aligned}
\frac{1}{2}\dot{r}^2 + \frac{h^2}{2r^2}\left(1 - \frac{2\mu}{r}\right) &= \frac{1}{2}c^2k^2 \\
\frac{1}{2}\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 + \frac{1}{2}u^2(1-2\mu u) &= \frac{c^2k^2}{2h^2} \\
\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + u = 3\mu u^2 &= \frac{3GM}{c^2}u^2
\end{aligned}$$

The Classical approximation, with $\mu \approx 0$, has the solution $bu = \sin\phi$, where b is the impact parameter; this is expected to be the asymptotic solution as $\phi \to \pi$ where the ray is assumed to come from. We multiply through by b and substitute the dimensionless $U = bu$:

$$\frac{\mathrm{d}^2U}{\mathrm{d}\phi^2} + U = \underbrace{\frac{3GM}{c^2b}}_{\beta} U^2$$

Writing $U(\phi) = \sin\phi + \beta U_1(\phi)$ and working first order to β, we find

$$\beta\frac{\mathrm{d}^2U_1}{\mathrm{d}\phi^2} + \beta U_1 = \beta\sin^2\phi \qquad \Rightarrow \qquad U_1'' + U_1 = \frac{1}{2} - \frac{1}{2}\cos 2\phi$$

$$\Rightarrow \qquad U_1(\phi) = A\sin\phi + B\cos\phi + \frac{1}{2} + \frac{1}{6}\cos 2\phi$$

$$= \frac{2}{3}\cos\phi + \frac{1}{2} + \frac{1}{6}\cos 2\phi$$

The constants are chosen to ensure that, going back in time to $\phi \to \pi$, asymptotically $U = \sin\phi$, meaning that $B = 2/3$ (so that $U_1(\pi) = 0$) and $A = 0$ (otherwise U would tend to $(1 + \beta A)\sin\phi$). We finally have

$$U(\phi) \approx \sin\phi + \beta\left(\frac{2}{3}\cos\phi + \frac{1}{2} + \frac{1}{6}\cos 2\phi\right)$$

$U \to 0$ as $\phi \to \pi$ as with $\beta = 0$, but due to the relativistic perturbation there is now another (small, negative) angle $-\Delta\phi$ at which this is true. On substitution and approximating $\cos\Delta\phi \approx 1$ and $\sin\Delta\phi \approx \Delta\phi$,

$$0 = -\Delta\phi + \frac{4}{3}\beta \qquad \Rightarrow \qquad \Delta\phi = \frac{4}{3}\beta = \frac{4GM}{c^2b}$$

as observed by Eddington and Campbell.

8 Cosmology

[Note: There are no formulae in the Relativity section of the formula book for this section, though some formulae are found in the Cosmology section of the formula book. These have been boxed in the usual way but be warned they are found in a different section of the formula book (and $a(t)$ is written $R(t)$; K as k).]

Cosmology applies only on very large scales, that is, when all the galaxy clusters etc are smoothed out.

8.1 Fundamentals

8.1.1 Definitions

- **Isotropic:** Looking in different directions, the Universe looks the same
- **Homogeneous:** As viewed from different positions at the same time[6], the Universe looks the same (for example, the density is the same)
- **Comoving:** Moving at such a velocity that everything else in the Universe moves isotropically, that is, moving at the "average" velocity of matter in the Universe

[6] Supposing the observers have synchronised their watches so they know what time to take measurements at

8.1.2 Fundamental Observers

There are some particularly useful reference frames with which to do Cosmology, called fundamental observers. We might imagine conceptually placing one at every point in 3D space. For a *synchronous* coordinate system:

- The fundamental observers have somehow synchronised their time coordinates $x^0 = t$. They travel along geodesics so this is also the proper time for each observer.
- The spatial coordinates x^i are constant in time for a given fundamental observer.

With this coordinate system, we can place some restrictions on what the observers will observe:

- Fundamental observers all observe the Universe to be homogeneous and isotropic
- They must therefore comove with matter in the Universe; if they were moving in a particular direction relative to the matter, then they would observe more matter coming towards them than away from them, breaking isotropy
- Their times must pass at the same rate; if one observer's time passed more quickly then there would have to be something special about their position, breaking homogeneity
- They must not be accelerating; if they were there would be a preferred direction (that in which the acceleration occurs). They thus travel along geodesics
- They must all measure the same density and it must be locally constant in space; otherwise there would be gradients and thus anisotropy

8.2 Robertson-Walker Metric

The line element in synchronous coordinates is given by

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 + g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$$

Now whatever the *spatial* behaviour of g_{ij}, its *temperal* behaviour must be the same at all points (homogeneity). It must thus be separable into $g_{ij} = -a(t)^2\gamma_{ij}(x^i)$, for some a (> 0 wlog) and γ_{ij}; the latter is the metric for 3D space. As such it must represent space as homogeneous and isotropic, which restrict it to the form

$$\mathrm{d}\sigma^2 = B(r)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$$

From this form, one can deduce the connection terms Γ^i_{jk} for the 3D space, from which one can deduce the components of the 3D $\boldsymbol{R}_{ijkl}$, $\boldsymbol{R}_{ij}$, and $\boldsymbol{R}$. After tedious calculation one obtains

$$\boldsymbol{R} = -\frac{2}{r^2}\left[1 - \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r}{B}\right)\right]$$

but for homogeneity this cannot depend on position, and so must be equal to a constant, taken traditionally to be $-6K$, with K the curvature constant. We hence obtain:

$$B(r) = \frac{1}{1 - Kr^2 + A/r}$$

for some integration constant A. A can be deduced by also using the fact that other invariants like $\boldsymbol{R}_{ij}\boldsymbol{R}^{ij}$ must be constants. This invariant is evaluated to be

$$\boldsymbol{R}_{ij}\boldsymbol{R}^{ij} = 12K^2 + \frac{3}{2}\frac{A^2}{r^6}$$

and so we must have $A = 0$; the 3D line element is then

$$\boxed{\mathrm{d}\sigma^2 = \frac{1}{1 - Kr^2}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2}$$

and the 4D Robertson-Walker metric is then

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a(t)^2\,\mathrm{d}\sigma^2$$

The behaviour of the space depends on K:

- $K = 0$: This simply corresponds to flat Euclidean space. Current measurements suggest that the Universe is like this
- $K > 0$: This describes a *closed* Universe, like a sphere
- $K < 0$: This describes on *open* Universe, like a hyperboloid

We now look at the non-Euclidean cases in turn:

8.2.1 Closed: $K > 0$

Changing coordinates $r \to \chi$:

$$r = S_K(\chi) \equiv \frac{1}{\sqrt{K}}\sin\left(\sqrt{K}\chi\right) \qquad \Rightarrow \qquad \mathrm{d}r = \cos\left(\sqrt{K}\chi\right)\mathrm{d}\chi$$

where χ runs from 0 to $\pi/\sqrt{K}$, has the same units as r, and is time-independent and hence a comoving coordinate. We then have

$$\mathrm{d}\sigma^2 = \frac{\cos^2\left(\sqrt{K}\chi\right)}{1 - \sin^2\left(K\chi\right)}\,\mathrm{d}\chi^2 + S_K(\chi)^2\,\mathrm{d}\Omega^2 = \mathrm{d}\chi^2 + S_K^2\,\mathrm{d}\Omega^2 = \mathrm{d}\chi^2 + S_K^2\,\mathrm{d}\theta^2 + S_K^2\sin^2\theta\,\mathrm{d}\phi^2$$

This also happens to the metric for the surface of a 3-sphere of radius $S_K(\chi)$. The volume of this closed space is finite:

$$V = \int_0^{\pi/\sqrt{K}}\mathrm{d}\chi\int_0^{\pi}\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi\,\sqrt{(1)(S_K^2)\left(S_K^2\sin^2\theta\right)} = 4\pi\int_0^{\pi/\sqrt{K}}\mathrm{d}\chi\,S_K^2 = \frac{4\pi}{K}\frac{\pi}{2\sqrt{K}} = \frac{2\pi^2}{K^{3/2}}$$

8.2.2 Open: $K > 0$

Using a different definition of S_K

$$r = S_K(\chi) \equiv \frac{1}{\sqrt{|K|}}\sinh\left(\sqrt{|K|}\chi\right) \qquad \Rightarrow \qquad \mathrm{d}r = \cosh\left(\sqrt{|K|}\chi\right)\mathrm{d}\chi$$

where now $\chi \in [0, \infty)$

$$\mathrm{d}\sigma^2 = \frac{\cosh^2\left(\sqrt{|K|}\chi\right)}{1 + \sinh^2\left(\sqrt{|K|}\chi\right)}\,\mathrm{d}\chi^2 + S_K^2\,\mathrm{d}\Omega^2 = \mathrm{d}\chi^2 + S_K^2\,\mathrm{d}\theta^2 + S_K^2\sin^2\theta\,\mathrm{d}\phi^2$$

which is also the metric for the surface of a 3-hyperboloid, the volumes of which are infinite.

8.3 Evolution of the Universe

For any value of K, we can write

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a(t)^2\left[\mathrm{d}\chi^2 + S_K^2\,\mathrm{d}\Omega^2\right] \qquad S_K(\chi) = \begin{cases} \sin\left(\sqrt{K}\chi\right)/\sqrt{K} & K > 0 \\ \chi & K = 0 \\ \sinh\left(\sqrt{|K|}\chi\right)/\sqrt{|K|} & K < 0 \end{cases}$$

Consider two observers at $\chi = 0$ and $\chi = \Delta\chi$ in some direction (θ, ϕ). The distance between them at a given time is $l(t) = a(t)\Delta\chi$. Define the quantity

$$\mathbf{H}(t) \equiv \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{1}{l}\frac{\mathrm{d}l}{\mathrm{d}t}$$

Depending on who you talk to, $\mathbf{H}(\text{now}) \equiv \mathbf{H}_0 \approx 70\text{km s}^{-1}\,\text{Mpc}^{-1}$, the *Hubble constant* (constant in space (for homogeneity) rather than time). This means that the observer at sees everyone else moving isotropically away equal fractional rates (i.e. twice as far away, twice the recession rate).

8.3.1 Cosmological Redshift

Suppose a photon travels from $(t_E, 0, 0, 0)$ to $(t_R, \chi_R, 0, 0)$ (angles taken to be 0 wlog from isotropy), so $x^\mu = (t(\zeta), \chi(\zeta), 0, 0)$ and $p^\mu = (p^0, \dot{\chi}, 0, 0)$. The photon follows a (null) geodesic, so follows a Lagrangian $\mathcal{L} = c^2\dot{t}^2 - a^2\dot{\chi}^2$. Using $e - \mathcal{L}\{\chi\}$ gives $a^2\dot{\chi} = \text{const.}$ so $\dot{\chi} \propto a^{-2}$. Being null, $c^2(p^0)^2 - a^2\dot{\chi}^2 = 0 \Rightarrow cp^0 \propto a^{-1}$. Now the energy, as before, is $E = g_{\mu\nu}u^\mu p^\nu$ where $\mathbf{u}$ is that of the observers – but the observers are not moving in space and hence $u^\mu = (1, 0, 0, 0)$ and $E = cp^0 \propto a^{-1}$. There will thus be a redshift:

$$1 + z = \frac{\lambda_R}{\lambda_E} = \frac{E_E}{E_R} = \frac{a(t_R)}{a(t_E)}$$

8.3.2 Friedmann Equations

The form of $a(t)$ is determined by EFE. The ingredients to this determination are

- Isotropy demands that ρ and p depend only on t, and hence that $T^{\mu\nu}$ is isotropic. We take the ideal fluid form
$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right)u_\mu u_\nu - p g_{\mu\nu}$$
- The observers' velocities are $u^\mu = (1, 0, 0, 0)$, being stationary
- The metric is of the form $\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a(t)^2\gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$ for the γ_{ij} discussed above
- A lengthy calculation using this metric shows that
$$R_{00} = 3\ddot{a}/a \qquad R_{0i} = 0 \qquad R_{ij} = -\frac{1}{c^2}\left(a\ddot{a} + 2\dot{a}^2 + 2Kc^2\right)\gamma_{ij}$$
- The EFE can be written as
$$R_{\mu\nu} = -\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) + \Lambda g_{\mu\nu}$$

The R_{00} and R_{ij} components together give

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{1}{3}\Lambda c^2} \qquad \boxed{\left(\frac{\dot{a}}{a}\right)^2 + \frac{Kc^2}{a^2} = \frac{8\pi G}{3}\rho + \frac{1}{3}\Lambda c^2}$$

known together as the Friedmann equations $\mathfrak{F}1$ and $\mathfrak{F}2$, which constrain a.

The conservation of energy equation, $\boldsymbol{\nabla}_\mu T^{\mu\nu}$, eventually gives

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0$$

Depending on the relationship between ρ and p, we find a relation between ρ and a. For dust, $p = 0$, which gives $\rho \propto a^{-3}$; for radiation, $\boxed{p = \rho c^2/3}$, which gives $\rho \propto a^{-4}$. Now ρ is the energy density $/c^2$, which for radiation is proportional to T^4, so also $\boxed{T^4 a^4 = \text{const.}}$.

8.3.3 $\Lambda = 0$

Recalling that $\mathbf{H} = \dot{a}/a$, the second Friedmann equation gives

$$K = \frac{a^2}{c^2}\frac{8\pi G}{3}\left(\rho - \frac{3\mathbf{H}^2}{8\pi G}\right)$$

so depending on the balance of ρ and $\mathbf{H}$, the Universe might be open, flat or closed.

For $\rho > 0$ and $p > 0$, $\mathfrak{F}1$ gives $\ddot{a} < 0$, so the scale factor was probably 0 once. Also suggests the age of the Universe is less than $a(\text{now})/\dot{a}(\text{now}) = 1/\mathbf{H}_0 \approx 14\text{Gyr}$.

For $K = 0$ (as seems to be observed), $\mathfrak{F}2$ gives relations between $(\dot{a}/a)^2$ and ρ, and ρ depends on a in a way depending on whether the Universe is dominated by matter or radiation. Either way, a proportionality between a and t, and hence a relation for $\mathbf{H}(t)$, can be found.

8.3.4 $\Lambda > 0$

The Universe is expanding, probably because $\Lambda > 0$. As the Universe expands, the density tends to 0. If $K = 0$, we then have $\mathbf{H} = \sqrt{\Lambda c^2/3}$ and $a = \exp\left(\sqrt{\Lambda c^2/3}t\right)$.